THEORY

OF THE

STRENGTH OF MATERIALS,

ILLUSTRATED BY APPLICATIONS

TO

MACHINES AND BUILDINGS.

BY

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PREFACE.

These notes having been originally prepared for the course of Machine and Mining Engineering, were during printing, somewhat expanded to accommodate that also of Civil Engineering, then introduced at the School of Mines. The arrangement of subjects is consequently not exactly what it will be made in another edition. I have introduced no tables, because, in view of new experiments being now prosecuted at the School of Mines and elsewhere upon American material, the old tables are becomming comparatively useless, and because, in numerous illustrations, I have given the value of the coefficients as they are used by the best builders, rendering this exposition of the theory of the strength of materials a safe and easy practical guide for the determination of strains and the proportioning of dimensions to support the same.

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INDEX.

PAGE

Molecular Forces 2
Compression: Extension 4
Shock 5
Flexion 8
Beam fixed at one end and the load uniformly distributed, 14
Beam resting on two supports with two loads 18
Beam fixed at both ends, load not uniformly distributed ... 25
Beam uniformly loaded and resting on three supports 26
Economizing material 28
Solids of equal resistance 34
Beam fixed at one end 38
Beam supported at two ends 40
Beam fixed at two ends 41
Beam supporting its own weight longitudinally 42
Bodies subjected to longitudinal and transverse strains 43
Horizontal beams 43
Vertical prisms 45
Empiric formula 53
Torsion 55
Shearing 60
Numerical Coefficients 62
Forged Iron 62
Steel 63
Cast Iron 63
Copper 64
Wood 65
Calculation of Thickness of a Boiler 66
Dimension of Pieces for Tubbing Shafts 72
Journals or Trunnions 76
Dimensions of the Teeth of Wheels 83
Fly Wheels 85
Balance Beam 88
Piston Rod 90
Connecting Rod 94
Crank 99
Axle of Fly Wheel, Water Wheel and Turbine 101
Pump Rod 103
Cross Strain on Axles for Water Wheels 105
Crane 109
Upright in a Wooden Dam or Lock Gate 111
Bottom of Lock, Masonry Apron 115

INDEX.

PAGE

Trusses 116
Two Rafters and Tie Beam 116
Beam Trussed by Ties and a Vertical Supporting Strut below, 122
Dimensions for Timber Bridges 123
Dimensions of Curved Beams 126
Lattice Truss 128
Dimensions of Voussoirs 130
Metallic Bridges 132
Suspended Bridges 136
Length of the Curve 141
Spandrel Fillings 147
Iron Truss 158
Purlins 160
Flooring 161
Distribution of Pressure on Surfaces in Contact 163
Plane Flexure in Curved Pieces 167
Graphic Method of Representing and Determining the Strain on any portion of a Truss 174
Po-t's 181
Whipple's 181
Bollman's 181
Fink's 181

THEORY

OF THE

STRENGTH OF MATERIALS.

RESISTANCE OF MATERIALS.

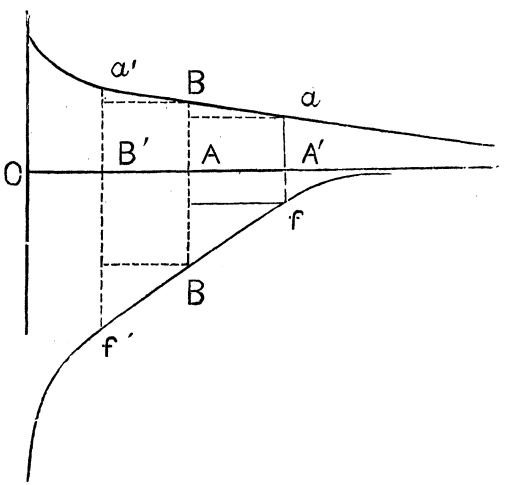
THE different parts of machines are made of pieces which under the efforts they receive must change more or less in form. Solid bodies are those which, up to a certain point, resist external impressions, and remain little variable in shape; but any force, however slight, exerted upon them, determines a certain material disadjustment, which at the same time however evokes internal reactions that tend to equilibrate the action of the forces.

If we take a rod and subject it to a traction, its molecules are separated by a small space, and this separation of molecules we say generates a certain resistance. How to conceive this development of force? It may be said that these molecules are in a state of equilibrium at certain distances apart, where there is equality between the attractive force (a quality inherent to matter), and the repulsive force—a quality equally inherent, even it be named heat. These attractive and repulsive forces do not vary in the same manner: the latter vary more markedly than the former; for by urging the molecules near together we can perceive that the repulsive forces increase

more rapidly than the attractive; and also by increasing the distance of the molecules apart we can observe that the repulsive forces diminish more rapidly than the attractive, since in both cases we experience an increasing resistance to our action. The efforts that may be exerted on matter are:

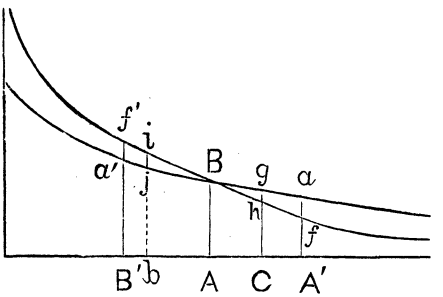
Compression; Extension; Flexion; and Torsion.

The point of departure of the calculations that have been made on the resistance of materials is the following principle: The resistance of a piece is, within certain limits, proportional to its actual state of compression or extension. That is to say, a force double, triple, etc. will be needed to elongate or compress a bar a double, triple, etc., quantity. A small graphic illustration will make this acceptable.



Imagine a rod of the length OA in molecular equilibrium. Let us represent by AB both the attractive forces and the repulsive forces, which are equal, since the rod is in the condition of equilibrium, and let us then consider an elongation AA'. The attractive forces will have diminished less rapidly than the repulsive, so that we shall represent by $A'f$ the repulsive forces and by $A'a$ the attractive forces. If we suppose the bar to have been shortened by a length

AB' the repulsive forces and the attractive forces will have increased, but the repulsive forces will have increased more than the attractive forces. Then $B'a'$ may represent the attractive forces, and $B'f'$ the repulsive forces. The curve $f'Bf$ will be the curve of the repulsive forces, $a'Ba$ will be that of the attractive forces.



If the points $A'B'$ are very near to the point A, these curves may be considered as right lines; and consequently it may be admitted that the increase of the repulsive over the attractive forces, and *vice versa*, will be proportional to the compression or elongation of the bar, for we shall have, in considering the similar triangles, Baf, Bgh, Bij, &c.:

$$\frac{af}{gh}=\frac{aB}{gB} \text{ or } \frac{af}{gh}=\frac{AA'}{AC}. \quad i.\ e.,\ gh : af :: AC : AA'.$$

We admit, then, that for bodies subjected to variable efforts, the deformation is proportional to the effort, within certain limits, and the force resisting the compression or the extension remains the same for the same deformation. When we do not exceed these limits we say that we have not surpassed the limits of elasticity. We shall consider the pieces of machines as subjected to forces which do not attain the limit of elasticity. A force ceasing to act, the body in this case returns purely and simply to its first position; and during such displacements the force is proportional to the increments or decrements of length which the body exhibits. If we have therefore a rod

of any section subjected to an effort of traction, its total elongation is proportional to the effort, since this principle of course holds for all the fibres of molecules which make up its mass.

If the effort tended to compress the rod, the shortening would be equal or proportional to the lengthening in the first case. If P then be the force applied, we shall have: $P = EA\frac{\lambda}{l}$; a formula in which E is a numerical coefficient; A the section of the bar; λ the elongation of the bar; and l the length of it.

P being the force that acts upon the body of section A, $\frac{P}{A}$ is the force distributed in the unit of section, we have $\frac{P}{A} = E\frac{\lambda}{l}$; whence we see that this force is proportional to $\frac{\lambda}{l}$; *i. e.*, to the ratio of the elongation to the length.

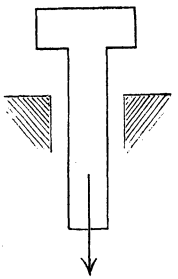
E is a numerical coefficient which varies with the nature of the body; it is called the modulus of elasticity, and is a large number, for if we make $A = 1$, $\lambda = l$, it becomes $P_1 = E$. That is to say, E would be the effort necessary to exercise upon a prism of 1^{m2} of section in order to double the length of it without breaking. For iron $E = 20{,}000{,}000{,}000$. If we call $\frac{\lambda}{l} = i$, we may write $P = EAi$.

Having i given, we may derive P, and reciprocally. By subjecting bodies to different forces, P, P^1, augmentations proportional to P, P', will result, which announce experimentally the value of i, for each. Among these elongations we shall note particularly those which cannot be increased without surpassing the limits of elasticity; for in general in the pieces of all constructions we endeavor not to exceed this limit, which we will call I. We shall have then to consider above all the quantity EI for different bodies. If we suppose that we have EI given we shall know for in-

stance that a prism of section A may be loaded to the limit of the weight AEI in units of weight.

Different from the resistance within elastic limits is the resistance to rupture R. This is the weight which must be loaded on a unit of section in order to cause absolute rupture. Having given $P^1 = AR$, a piece in practice ought to be loaded with only a fraction of AR. If x be this fraction, $\frac{1}{4}$, $\frac{1}{6}$, or $\frac{1}{10}$, according as the member is more or less important, or according to the chances of overcharge to which it is exposed, we call x the coefficient of security. The coefficient varies with the homogeneity of the piece. Generally x is made equal to $\frac{1}{10}$. In order to acquire a positive idea of the meaning and value of the coefficient of security, we must regard whether it be applied in any case to the limit of elasticity or rupture. In this latter view it should be made smaller than in the former, for the inconveniences resulting from rupture are more to be feared than those proceeding from exceeding the limit of elasticity.

We have often to consider a condition in which bodies are subjected to the action of masses in movement. They then have to resist a certain quantity of living force—as for instance when a body falls on another, producing a shock.



Let us consider a vertical rod made with a shoulder or rimbase on which it rests, or, rather, on which it may fall. During the elongation after a fall, there will be a work produced, which will reduce and annul

the living force of the body. We have, at the moment of the utmost elongation :

$$P = EAi, \quad i = \frac{\lambda}{l}$$

If we consider the prism from the moment at which it has the length l up to the moment when it has the length $l+\lambda$, for an increase $d\lambda$ we have :

$$P.d\lambda = Eai.d\lambda \quad \text{But } i = \frac{\lambda}{l}$$

Consequently : $\lambda = li$, $d\lambda = ldi$, and

$$Pd\lambda = EAildi = EAlidi$$

But Al is the volume V. Therefore :

$$Pd\lambda = EVidi$$

If we consider the bar from the elongation 0 up to an elongation λ, and integrate, we have :

$$\int_0^i EVidi = \tfrac{1}{2}Ei^2 V = \int Pd\lambda$$

And if we substitute for i the proportional elongation I, corresponding to the limit of elasticity, we have :

$$\int Pd\lambda = \tfrac{1}{2}EI^2 V$$

$\frac{1}{2}EI^2 V$ is then the living elastic resistance of the prism. Thus while the resistance to an effort depends upon the section, the living elastic resistance depends upon the volume of the prism ; or again, while $\frac{1}{2}EI^2$ is the measure in which a material can resist shocks, EI serves to estimate its opposition to an effort, each within elastic limits. A body may have a limit of elasticity greater and a coefficient of resistance smaller than the quantities analogous for another body, according as E is large or small comparatively to I; or again, in the same body EI may be large, and EI^2 small, and *vice versa*. Thus in castings, EI is large, but $^1EI^2$ is comparatively small. For annealed iron wire, EI is less than for unannealed wire, but $\frac{1}{2}EI^2$ is larger. Hence, for instance, it results that annealed iron wire should be used in the construction of a suspended bridge which is sub-

jected to shocks. In order to decide, therefore, what material should be employed in the construction of a piece, we must foresee whether the piece is to be subjected to the action of shocks as well as to pressure and traction. If now a body be examined from an elongation null, up to that which produces rupture, and the question be to integrate : $\int P.d\lambda.$, we know that in the first moments P is proportional to EAI, but near rupture not being acquainted with the laws that regulate the deformations, we can no longer state $P = EAI$. However, P is always proportional to the section, and we may place : $P = Af(i)$

Wherefore we must determine : $\int Af(i)d\lambda$ But

$$i = \frac{\lambda}{l} \qquad \lambda = il. \qquad d\lambda = l.di$$

And we may seek $\int_0^I Af(i).l.di = \int_0^I Alf(i)di$

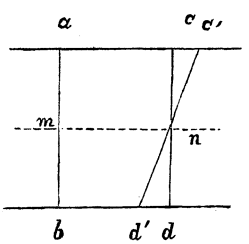
Or $\int_0^I Vf(i)di = V\int_0^I f(i)di$

And supposing this integral known, we may say that $\int_0^I f(i)di$ is the coefficient of the living resistance to rupture, and $V\int_0^I f(i)di$ is the living resistance to rupture.

When permanent loads alone are in question, we only have to consider static resistances to rupture, but if there are shocks, living resistances also.

BODIES SUBJECTED TO EFFORTS OF FLEXION.

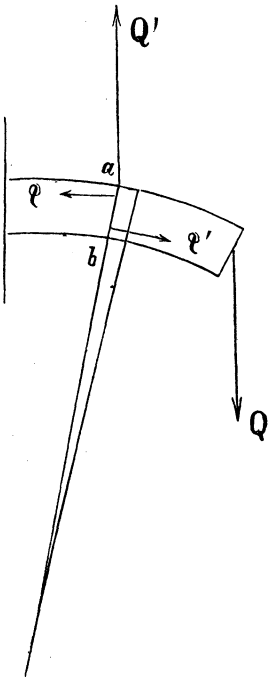
If we consider a prism fixed at one end and loaded at the other with a certain weight, the axis of the prism we know will bend. Let us suppose the prism symmetrical with reference to a vertical plane passing through its centre of gravity. If a force Q be applied in this plane, the prism will be deformed without issuing from the plane of symmetry, and all being alike in reference to this plane, all components will have resultants situated in this plane, so that we shall reason as if everything took place in the same. If



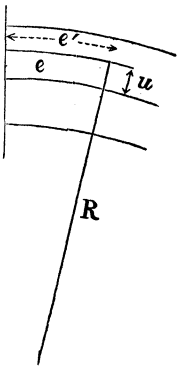
we consider two straight sections *abcd* infinitely near together and parallel before the action of the force Q, they will no longer be parallel when this force shall have operated. The upper fibres will be lengthened, the lower shortened, and there will appear one, *mn*, which will not have changed, called neutral fibre.

In any straight section *ab* there will exist a vertical force Q' opposing itself to the rupture of the prism by shearing. There will also be in the upper part, forces φ which tend to draw back the severed molecules, and in the lower part, likewise, forces φ' tending on the contrary to repel apart the molecules forced together. The forces φ and φ' are parallel.

The forces Q and Q' are also parallel. We have then for equilibrium : $Q = Q'$ and : $\Sigma\varphi = \Sigma\varphi'$



The whole of the forces φ and φ' form a couple that destroys the couple $Q\ Q'$. Let us consider first the neutral fibre having a radius of curvature R, and note then a fibre situated a distance above it $= u$; *i. e.*, having a radius of curvature $R+u$. This fibre



will be elongated, and if its length become e', the length of the neutral fibre being e, we shall have $e' : e :: R+u : R$; whence $(e'-e) : e :: u : R$, and $\frac{e'-e}{e} = \frac{u}{R}$.

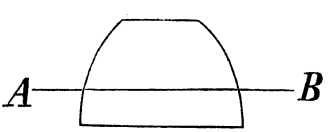
$\frac{Eu}{R}$ is the force determining the elongation $\frac{u}{R} = \frac{e'-e}{e}$ or $\frac{\lambda}{l}$; and $\frac{Eu}{R}\omega$ is the amount of that force distributed on the section ω. This is, then, the value of one of the forces φ; and therefore:

$$\Sigma\varphi = \Sigma\frac{Eu}{R}\omega = \frac{E}{R}\Sigma u\omega$$

For the shortened fibres we should have an analogous formula:

$\Sigma\varphi' = \frac{E}{R}\Sigma u'\omega'$ Now: $\Sigma\varphi = \Sigma\varphi'$ Wherefore: $\Sigma\omega u = \Sigma\omega' u'$ But $\Sigma\omega u$ is the moment relatively to the neutral axis of the elements above it; and $\Sigma\omega' u'$ is the corresponding moment of the elements below; and if $\Sigma\omega u = \Sigma\omega' u'$, it follows that the neutral fibre passes through the centre of gravity of the entire section, since this relation exists only for the centre of gravity. Consequently, the neutral fibres are in a surface of horizontal directrix passing through the centres of gravity of all the sections under consideration. Having the position of the neutral fibre, and knowing φ, the moment of the couple $\varphi\ \varphi'$ is the sum of the moments of these forces relative to the neutral axis. The expression for any such elementary moment is:

$E\frac{u}{R}\omega.u = \frac{E\omega u^2}{R} = \frac{E}{R}\omega u^2$, and consequently the sum of the moments φ will be: $\frac{E}{R}\Sigma\omega u^2$. Now $\Sigma\omega u^2$ is the moment of inertia of the section of the prism rela-

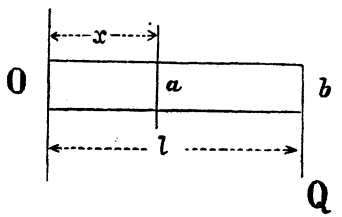


tively to the line AB passing through its center of gravity; and calling it μ, we have: $\frac{E}{R}\Sigma\omega u^2 = \frac{E\mu}{R}$

Such is the moment of elasticity of any section, and it is necessary that this be equal to the moment of the forces Q, since there is equilibrium among the forces Q and φ.

Therefore: $\frac{E\mu}{R} \quad \Sigma MQ$ $\qquad (A)$

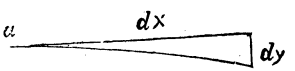
This is the equation that serves, near enough for practice, to resolve all the different problems that are usually presented concerning the resistance of materials.



Let us first consider a prism fixed at one of its extremities and loaded at the other, and observe a section at the distance x from the fastening. We have, by applying equation (A):

$$\frac{E\mu}{R} \quad Q(l-x)$$

For it is evident that the moment of Q relative to the point a, is $Q(ab)$, or $Q(l-x)$.



To discuss this equality, we have a known general value for the radius of curvature:

$$R = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)}$$ But $\frac{dy}{dx}$ is very small in equation A, for dy is the vertical depression below a of the point next to a. When the beam is but slightly bent, dy is almost null compared to dx. The relation

$\frac{dy}{dx}$ tends towards zero, and indeed it becomes zero when the force Q is zero. We may, then, in order to simplify, suppose $\frac{dy}{dx} = 0$, and state:

$R = \frac{1}{\frac{d^2y}{dx^2}} = \frac{dx^2}{d^2y}$. Replacing R by its value in the equation $\frac{E\mu}{R} = Q(l-x)$, this becomes:

$E\mu.\left(\frac{d^2y}{dx^2}\right) = Q(l-x)$. This equation is the more nearly exact as the inflexion of the bar is less. Integrating, we obtain:

$$E\mu\frac{dy}{dx} = Q\left(lx-\frac{x^2}{2}\right) + \text{constant.} \quad (a)$$

Now as there is fixation at $x = 0$, the tangent to the curve of the neutral axis is horizontal there, or $\frac{dy}{dx} = 0$. If we place these values of the variables in equation (a), we shall see that the constant is 0, and by resuming (a) and integrating again, we obtain the equation of the neutral axis:

$$E\mu y = Q\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + \text{constant.}$$

Here remarking that $x = 0$, for $y = 0$, we see that the constant is again 0, and there remains:

$$E\mu y = Q\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$$

If we designate the versin of the entire curve by f we shall have: $E\mu f = Ql^3(\frac{1}{2}-\frac{1}{6})$, for the versin f is the quantity that y becomes when $x = l$.

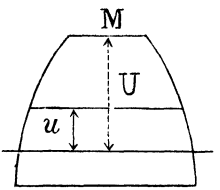
$$E\mu f = Q\frac{l^3}{3}$$

$$f = \tfrac{1}{3}\frac{Q}{E\mu}l^3$$

The versin then is proportional to the cube of the length of the piece.

We may ask what effort the prism can support without the strain on any point exceeding a determined limit. Say then that in any section whatever

we have given: $\frac{u}{R}$, the measure of the elongation or shortening of any fibre situated at a distance u from the neutral axis. The strain is maximum where the elongation is maximum, at the point M, then, since this point is the most distant from the neutral line.



Let us represent by U the maximum ordinate, and determine the expression for the strain $E\frac{U}{R}$. From equation (A), $\frac{E\mu}{R} = \Sigma MQ$, we derive, multiplying by $\frac{U}{\mu}$; $\frac{EU}{R} = \frac{\Sigma MQ}{\mu} U$.

And this is the measure of the maximum fatigue. In each section, the upper or the lower fibre will be most strained, and if we pass from one section to another, the strain will be maximum for that in which the moment of Q is maximum. Therefore in a prism confined at one of its extremities, the section confined is most strained, for here MQ is maximum. In this section, then, rupture will take place, either by crushing or by tearing, according as the upper part is at a less or greater distance from the centre of gravity than the lower. Now $E\frac{u}{R}$, the general expression for strain, is the force φ, and for the section confined this force must have a maximum value: $\Sigma\frac{MQ}{\mu}U = \Phi$, which becomes, in the case considered: $\Phi = \frac{Ql}{\mu}U$. If we would strain this section to the limit of elasticity, we must have $\Phi = EI$, and consequently:

$$EI = \frac{Ql}{\mu} U. \quad (b)$$

If the prism is given, all the quantities are given except Q, and we may derive from the equality (b):

$$Q = EI\frac{\mu}{lU}$$

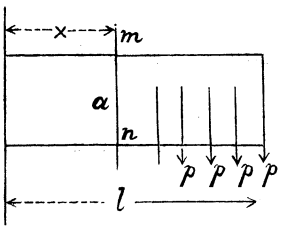
In order that the prism may have the greatest possible useful resistance, it must be fashioned so as to admit Q as great as possible. We must then render μ as great as practicable, and U as small, or make $\frac{\mu}{U}$ maximum, which amounts, in short, to making μ a maximum, for μ varies as the square of analogous di-dimensions, while U varies directly as these same.

If, instead of taking the equation last used, we take one of the form $Q = \varkappa EI\frac{\mu}{lU}$, we may consider this as giving a load which the prism will support having regard to a certain coeffieient of security, $= \varkappa$.

If now we would determine the versin corresponding to the limit of elasticity, we must replace Q in the formula which gives the versin by the preceding value.

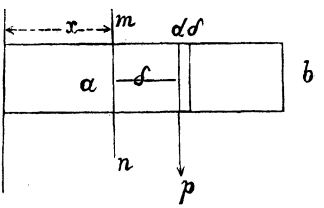
Let us next consider a prism fixed at one of its extremities, the load being uniformly distributed over the prism. Let p be the load per linear metre, and let us apply equation (A):

$$\frac{E\mu}{R} = \Sigma MQ$$



Let l be the length of the prism, and mn a section situated at a distance x from the fastened section. ΣMQ is the sum of the moments of the forces p rela-

tive to the point a. We have to calculate only the forces on the right of any section mn, for these are the only ones which tend to cause rupture there; all on the left serve only to fix that part. Let us assume an element situated at a distance δ from the point a.



This element having a length $d\delta$, the force acting upon it will be $pd\delta$, since it is p per unit of surface. The moment of this force will be $p\delta d\delta$, and the sum of the moments will be the integral of this expression from the point a as far as the point b; *i. e.*, from $\delta = 0$ to $\delta = (l-x)$.

Then $\Sigma MQ = \int_0^{(l-x)} p\delta d\delta = p\int_0^{(l-x)} \delta d\delta$

Now $\int \delta d\delta = \frac{\delta^2}{2}$ Then $\int_0^{l-x} \delta d\delta = \frac{(l-x)^2}{2}$

and $\Sigma MQ = \frac{p}{2}(l-x)^2$.

This equality is otherwise evident, by observing that the resultant of all the forces p applied from the point a to the point b has for value $p(l-x)$, and that this resultant acts at the middle of ab; *i. e.*, at a distance $\frac{l-x}{2}$ from the point a. Now the moment of the resultant being equal to the moments of the components, it suffices, in order to have the moments of the components, to estimate the moment of the resultant, and this moment is:

$p(l-x)\frac{l-x}{2} = \frac{p}{2}(l-x)^2$, as before.

Therefore $\frac{E\mu}{R} = \frac{p}{2}(l-x)^2$, or,

$$E\mu\frac{d^2y}{dx^2} = \frac{p}{2}(l^2-2lx+x^2)$$

Integrating, $E\mu\frac{dy}{dx} = \frac{p}{2}\left(l^2x-lx^2+\frac{x^3}{3}\right)$

The constant is null, because the prism is enclosed at its extremity, which amounts to saying that $\frac{dy}{dx} = 0$, for $x = 0$. Integrating again, we have:

$$E\mu y = \frac{p}{2}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right)$$

The constant is still null here, for we have $y = 0$, for $x = 0$. In order to obtain the versin, we must replace x in the value of $E\mu y$ by l, and the value of y will be the versin. Designating it by f', we derive:

$$E\mu f' = \tfrac{1}{2}pl^4(\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{12})$$
$$E\mu f' = \tfrac{1}{8}pl^4 = \tfrac{1}{8}pl.l^3$$

Now pl is the total load that the prism supports. If we designate it by Q, we have:

$E\mu f' = \frac{1}{8}Ql^3$ Whence: $f' = \frac{1}{8}\frac{Q}{E\mu}l^3$.

This versin is three-eighths of that found when we supposed the force to be applied at the extremity of the prism. Let us now examine the strain.

$$\frac{EU}{R} = \frac{\Sigma MQ}{\mu}.U = \frac{p(l-x)^2}{2\mu}U.$$

The sections being identical, the maximum will take place in the section for which $x = 0$, and we have:

$$\frac{EU}{R} = \frac{pl^2}{2\mu}U$$

If we wish the fatigue to be equal to a given quantity $\varkappa EI$, we shall state: $\frac{pl^2}{2\mu}U = \varkappa EI$, and $pl = 2\varkappa EI\frac{\mu}{lU}$. But pl is the entire load supported by the prism; if we designate it by Q', we obtain $Q' = 2\varkappa EI\frac{\mu}{lU}$. This is double the load Q which could be placed entire at one end of a prism confined at the other. Thus the same prism in the same conditions may sustain a double weight, if this weight, instead of being applied at its end, is distributed symmetrically over all its length.

The results of an arrangement on a beam in both these ways at once may be found easily. The versin and the strain are in short the sum of the versins and the strains found in treating the two cases separately.

In the cases we have thus studied, the elastic static strength of a beam was alone considered for a limit. We may, however, often be required to regard its living elastic resistance. If then we suppose a load Q suddenly applied determining a versin f, we must consider the resistance of the beam as equivalent to an expression of the form :

$$\int Qdf$$

But between Q and f the relation within elastic limits is :

$$f = \tfrac{1}{3}\frac{Q}{E\mu}l^3$$

Wherefore

$$\int Qdf = \int \frac{3E\mu}{l^3} fdf = \frac{3E\mu}{l^3}\int fdf$$

Taking this integral from 0 to the value of f injuring elasticity, it will become :

$$\int_0^f Qdf = \tfrac{3}{2}\frac{E\mu}{l^3}f^2$$

And if in this expression we replace f by its equivalent in terms of Q, we shall find :

$$\int_0^f Qdf = \tfrac{3}{2}\frac{E\mu}{l}\tfrac{1}{9}\frac{Q^2}{E^2\mu^2}l^6$$

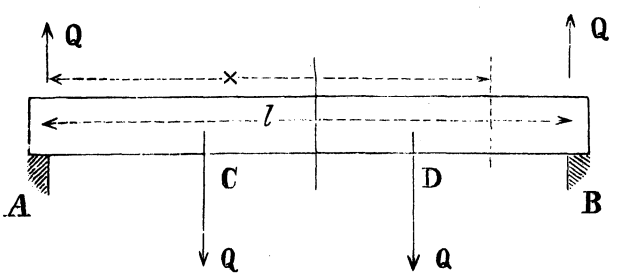
$$\text{or} \quad \int_0^f Qdf = \tfrac{1}{6}\frac{Q^2}{E\mu}l^3$$

If now we apply a coefficient, and take Q within the limit of elasticity we know we have :

$Q^2 = x^2 E^2 I^2 \frac{\mu^2}{l^2 U^2}$; replacing which value of Q^2 in the expression of resistance $\frac{1}{6}\frac{Q^2}{E\mu}l^3$, this last becomes : $\frac{1}{6}\frac{x^2 EI^2 \mu l}{U^2} = \frac{1}{6}x^2 EI^2 \frac{\mu}{U^2} l$. We see then that living elastic resistance is proportional to the length

of a prism, as well as to its section. The prism, in fact, will resist a shock the better in proportion to its length, because it is capable of bending more. As an illustration: when rails are made they are subjected to certain trials before being put in the market—especially to loads and shocks; for a length then of 1^m., between two supports, the rail ought, say, to support 15,000kg.; and bear the shock of a ram of 300kg. falling 1^m. This last condition gives immediately the value of $M\frac{V^2}{2} = 300$, which must be put equal to $\frac{1}{6}x^2 EI^2 \frac{\mu}{U^2} l$, in order to determine the sectional area of the rail for living resistance.

Let us now consider the case of a prism resting on two supports and subject to the action of two loads placed symmetrically, as at C and at D. The same equation will evidently not serve for a point between B and D and another between C and D. Let us then first examine a point situated between B and D, and apply formula A.



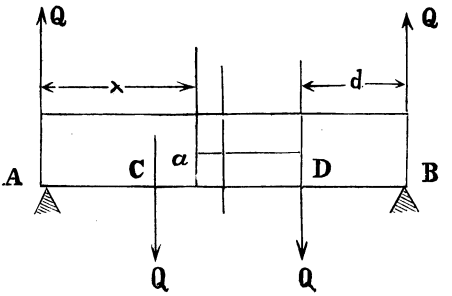
Recalling always that ΣMQ designates the sum of the moments of all the forces situated to the right of the point considered, reckoning the x from the left, it is plain that ΣMQ here is only the moment of the reaction Q applied at B, and as this force acts upward, its sign and its moment are negative.

Therefore, between B and D we have:

$$E\mu\frac{d^2y}{dx^2} = -Q(l-x)$$

and the question from this departure would be treated

like the preceding ones. We see that when $x = l$, $R = \infty$.



Turn then now to the points comprised between C and D. Here it will be necessary to introduce the moments of the forces applied at D and at B. Consequently we shall have :

$$E\mu\frac{d^2y}{dx^2} = -Q(l-x) + Q(l-x-d)$$

designating by d the distance $AC = BD$, wherefore :

$$E\mu\frac{d^2y}{dx^2} = -Q(l-x) + Q(l-x)-Qd$$

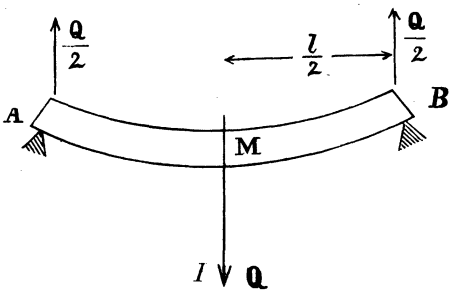
or finally,

$$E\mu\frac{d^2y}{dx^2} = Qd$$

This value of $\frac{E\mu}{R}$ is then a constant quantity for all the points between C and D, and, therefore, for all these points R is constant, and consequently the curve of the neutral fibre between C and D is an arc of circle. Between BD and CA the neutral axis is a parabola of the third degree, according with the arc of circle from C to D. Analytic continuity does not exist between these curves, since several equations are necessary to represent the locus of the neutral fibre. If instead of only two forces a much greater number were applied, this locus would become a series of curves according with each other at points where their radii of curvature were the same. The value of the versins can easily be determined.

If the points C and D approached each other, and finally coincided in the middle of CD, one equation

only would be necessary from the point A to the point M, middle of AB, and another from the point M to the point B. The locus of the neutral points would be two portions of two symmetrical parabolas; different ones, however, and not two consecutive arcs of the same curve. We may therefore consider the



case as if the solid were fixed at its middle and influenced at its two extremities each by a force tending to lift it, for the portion MB for instance is indeed in the same condition as if it were fastened at M, where the tangent is horizontal. A beam, then, supported at its two ends and bearing a weight in the middle, may be treated as if it were fastened at the middle, and loaded at the ends with weights each half of that at the middle. The strain and the versin will be the same for each half of l. But we have, as we know, in this supposition, for the general expression of versin:

$$f = \tfrac{1}{3}\frac{Q}{E\mu}l^3$$

and for the maximum strain:

$$\varkappa EI = \frac{Ql}{\mu}U$$

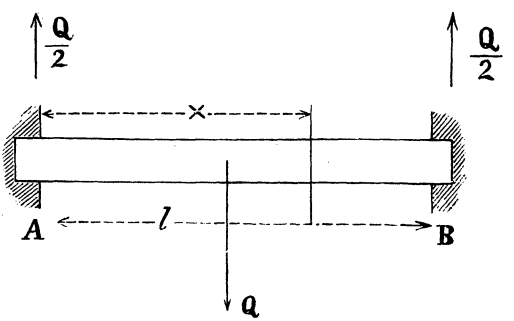
We will then in these values replace Q by $\frac{Q}{2}$, and l by $\frac{l}{2}$, and derive for the versin in this particular case:

$$f = \tfrac{1}{16}\frac{Q}{3E\mu}l^3 = \tfrac{1}{48}\frac{Q}{E\mu}l^3$$

And for the strain :

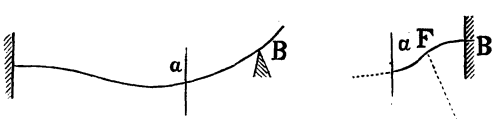
$$\varkappa EI = \tfrac{1}{4}\frac{Ql}{\mu}U$$

If therefore we consider the same prism successively as first confined at one of its ends and loaded at the other, and then supported at both ends and loaded at the middle with the same weight, we find that the versin in the second case will be one-sixteenth of the versin in the first, and the strain will be four times less in the second case than in the first; whence it results that the same prism thus supported may sustain a load four times greater than when fixed only at one end.



Let us now consider where a body is confined at its two ends, as is often done with beams, because if a prism be not fastened, it bends in such manner that a tangent at an extremity is not horizontal. In any section, then, situated at a distance x from the left end, the moment of elasticity is:

$$E\mu\frac{d^2y}{dx^2} = \Sigma MQ + A = -\frac{Q}{2}(l-x) + A$$



The term A is a constant, the aim of which is to represent the effect of the fastening. It is truly a constant, for this effect we can but suppose is the

same on all the sections. If now, moreover, the prism were free at B, it would be convex all along from a to B; while if it be fixed there its convexity must disappear somewhere, and be replaced by concavity near that point, in which vicinity, consequently, there will exist a point F of inflection. We have then, to resume:

$$E\mu\frac{d^2y}{dx^2} = -\frac{Q}{2}(l-x)+A$$

Integrating: $E\mu\frac{dy}{dx} = -\frac{Q}{2}\left(lx-\frac{x^2}{2}\right)+Ax+C$

The constants A and C are to be determined. To this end we will remark that if we make $x=\frac{l}{2}$, we must have, on account of symmetry, $\frac{dy}{dx}=0$, for the tangent is horziontal at this point. Moreover, the effect of confinement gives, for $x=l$, $-\frac{dy}{dx}=0$. By introducing these conditions A and C may be evaluated. For $x=\frac{l}{2}$ we get $-\frac{Q}{2}[l^2(\frac{1}{2}-\frac{1}{8})]+A\frac{l}{2}+C=0$

and for $x=l$, $-\frac{Q}{2}l^2(1-\frac{1}{2})+Al+C=0$

Or indeed $-\frac{3}{16}Ql^2+\frac{Al}{2}+C=0$

And $-\frac{1}{4}Ql^2+Al+C=0$.

These two equations determine A and C. Subtracting, first the one from the other, we obtain:

$$\frac{Al}{2}-\frac{1}{16}Ql^2=0$$

$$\frac{Al}{2}=\frac{1}{16}Ql^2 \qquad A=\frac{Ql}{8}$$

and substituting this in one of the primitive equations, the second, for example, we may find:

$$C=\frac{1}{4}Ql^2-\frac{1}{8}Ql^2 \qquad C=\frac{1}{8}Ql^2$$

Replacing, then, these values of A and C in the equation which gives $\frac{dy}{dx}$, we shall deduce:

$$E\mu\frac{dy}{dx}=\frac{Ql^2}{8}-\frac{3}{8}Qlx+Q\frac{x^2}{4}$$

and integrating:

$$E\mu y = \frac{Ql^2}{8}x - \tfrac{3}{16}Qlx^2 + \frac{Qx^3}{12} + C'$$

in which last, C' will be determined by the condition that $y = 0$ for $x = l$, which gives:

$$E\mu y = 0 = Ql^3(\tfrac{1}{8} - \tfrac{3}{16} + \tfrac{1}{12}) + C'$$

$$0 = \tfrac{1}{48}Ql^3 + C' \qquad C' = -\frac{Ql^3}{48}$$

Consequently, replacing all in the value $E\mu y$, we have:

$$E\mu y = \frac{Ql^2}{8}x - \tfrac{3}{16}Qlx^2 + Q\frac{x^3}{12} - Q\frac{l^3}{48}$$

In this example, however, there is evidently again a locus formed by two identical curves, for the results just found only relate to the right half of the prism, and the whole neutral line cannot be represented by a single equation. In order, however, to obtain the versin, we need only in this equation make $x = \frac{l}{2}$, which will give:

$$E\mu f = Ql^3(-\tfrac{1}{48} + \tfrac{1}{16} - \tfrac{3}{64} + \tfrac{1}{96})$$

$$E\mu f = \frac{Ql^3}{3}\tfrac{1}{64}$$

When the solid was fastened at one end and loaded at the other with a weight Q we found;

$$f = \tfrac{1}{3}\frac{Q}{E\mu}l^3$$

When the solid was placed upon supports and bore at its middle a load Q we found:

$$f_2 = \tfrac{1}{3}\frac{Q}{E\mu}l^3 \cdot \tfrac{1}{16}$$

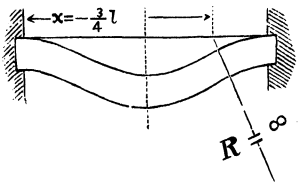
And now that the solid is supposed confined at its extremities we find:

$$f_3 = \tfrac{1}{3}\frac{Q}{E\mu}l^3 \cdot \tfrac{1}{64}$$

Whence: $\qquad f_3 = \tfrac{1}{4}f_2 = \tfrac{1}{64}f$

Remembering always that the strain is given by the formula:

$$\frac{Eu}{R} = \frac{E\mu}{R} \cdot \frac{u}{\mu}$$



We have in this case :

$$\frac{E\mu}{R} = -\frac{Q}{2}(l-x)+\frac{Ql}{8} = \frac{Q}{2}\left(\frac{l}{4}+x-l\right) = \frac{Q}{2}(x-\tfrac{3}{4}l)$$

and consequently : $E\frac{u}{R} = \frac{Q}{2\mu}(x-\tfrac{3}{4}l)u$

If in this expression we make $x = \frac{3}{4}l$, the strain becomes : $\frac{Eu}{R}=0$; whence $R = \infty$; here, then, is the point of inflection, where there is no curvature, or where, in short, the curvature and strain both pass through zero, as must be the case, since before inflection the curve is in one direction, and the fibres are stretched in one direction ; while after they are stretched and curved in a contrary manner. Let us now proceed to seek where the point of rupture will be. If we make $x = l$ in the expression for strain we shall find one maximum :

$$\frac{Q}{8\mu}lU = \frac{EU}{R}$$

and if also we make $x = \frac{l}{2}$ the same numerical value of strain is exhibited at that point :

$$-\frac{Q}{8\mu}lU = \frac{EU}{R}$$

The fatigue is then the same in the section fastened and in the middle section ; the body tends, therefore, to break equally at either.

The value of the pressure which can be borne within elasticity is :

$$\varkappa EI = \frac{Q}{8\mu}lU = F_3$$

When the body was supported at its two ends we found :

$$\varkappa EI = \frac{Q}{4\mu}lU = F_2$$

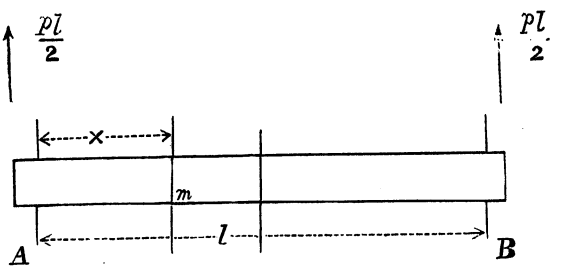
When the body was supported at one end and bore a weight Q at the other, we found:

$$\varkappa EI = \frac{Q}{\mu} l U = F_1$$

We have, then: $F_3 = \frac{1}{2}F_2 = \frac{1}{8}F_1$
and therefore see that a given prism fixed at its two ends and loaded in the middle, will bear without breaking a weight eight times greater than if it were simply confined at one end and the weight in question applied at the other.

In practice loads are to be calculated as if the pieces were not confined at all, then if these be fastened down there is obtained a great margin of security.

Now consider the case in which a load is uniformly distributed upon a prism fixed at its two ends, and



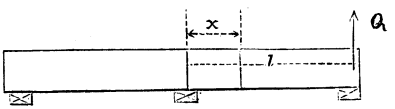
let us discover ΣMQ. We have therein firstly all the forces p which have a resultant $p(l-x)$ applied to the middle of mB; *i. e.*, to a distance from m equal to $\frac{1}{2}(l-x)$, and the moment of these forces p is $\frac{1}{2}p(l-x)^2$. We have afterwards the force $\frac{1}{2}pl$ applied at B, the moment of which is $\frac{p}{2}l(l-x)$, and finally, the moment due to the fastening, which we shall designate by A, so that we shall have:

$$E\mu\frac{d^2y}{dx^2} = \frac{p(l-x)^2}{2} - \frac{pl}{2}(l-x) + A$$

There will evidently always exist a convex part terminating in a concave one. The calculation is identical with the preceding; the versin is reduced

to one-fourth, and the strain to two-thirds. We may observe that the effect of fastening when the load is uniformly distributed is less than in case of one force applied at the middle of the prism.

BEAM UNIFORMLY LOADED AND RESTING ON THREE SUPPORTS.



If the tangent at the middle be horizontal, as is

the intention, then this beam may be considered as if each half were fixed at A, and supported free at the end. We shall have:

$$E\mu\frac{d^2y}{dx^2}=\frac{p}{2}(l-x)^2-Q(l-x)$$

Q is not at present known, for the three supports cannot be supposed to bear the entire pressure equally. Developing and integrating once, we find:

$$E\mu\frac{dy}{dx}=\frac{p}{2}\left(l^2x-lx^2+\frac{x^3}{3}\right)-Q\left(lx-\frac{x^2}{2}\right)$$

The constant is evidently null.

Integrating again, we obtain:

$$E\mu y=\frac{p}{2}\left(l^2\frac{x^2}{2}-l\frac{x^3}{3}+\frac{x^4}{12}\right)-Q\left(l\frac{x^2}{2}-\frac{x^3}{6}\right)$$

This equation serves to determine Q, for the relation subsists that $y=0$ when $x=l$. Introducing this consideration, we shall find $Q=\frac{3}{8}pl$. Therefore, each extreme support bears $\frac{3}{8}$ of half the whole load, or $\frac{3}{16}$ of all, and consequently the middle support bears $\frac{10}{16}$ of all.

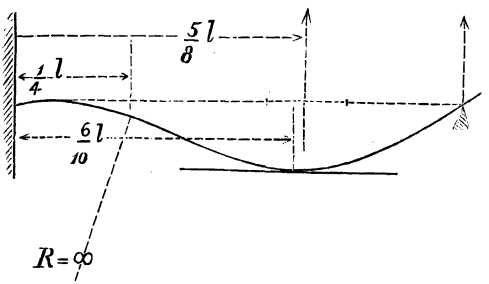
If now, in the general equation for $E\mu\frac{d^2y}{dx^2}$ we substitute the value for Q just found, we obtain:

$$E\mu\frac{d^2y}{dx^2} = \frac{p}{2}(l-x)^2 - \tfrac{3}{8}pl(l-x)$$

$$= (l-x)\frac{p}{2}(l-x-\tfrac{3}{4}l) = \frac{p}{2}(l-x)(\tfrac{1}{4}l-x)$$

This is the moment of all the forces, and being proportional to the strain, this last will be greatest where that expression is maximum. The algebraic maximum of the expression is a minus quantity, and refers to the maximum of the upward strain.

It takes place at $x = \frac{5}{8}l$. The maximum positive



strain in the problem is at $x = 0$; this last is greater than the other, being $\frac{1}{8}Wl$, while the first is $-\frac{1}{14}Wl$ approximately. The point of inflection is found by making $\frac{dx^2}{d^2y} = 0$; it is at $x = \frac{1}{4}l$. The maximum minus strain is then half way between the point of inflection and the free end. If $\frac{dy}{dx}$ obtained from a first integration be made $= 0$, the position of the horizontal tangent is found at $x = \frac{6}{10}l$. The versin is then $0.005\frac{Wl^3}{E\mu}$

We operate in this same manner for other similar cases; if there are numerous curves according with each other, it is necessary to determine for every curve the point where there is the greatest tendency to rupture, and discover finally the maximum of all these maxima. In this way we expose the point where rupture is likely definitely to take place.

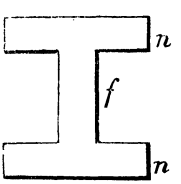
We will not treat of other cases at present, but immediately pass to an important question in practice, which is to ascertain *the manner in which material should be arranged, in order to employ it as economically as possible.*

We found: $\varkappa EI = \Sigma MQ\frac{U}{\mu}$. For economy of material. then, the forces Q being given, $\varkappa EI$ must be made a minimum; or $\varkappa EI$ being given, we endeavor to augment Q as much as possible. Now, μ is the sum of quantities ωu^2, $\mu = \Sigma\omega u^2$, and $\Sigma\omega u^2 = \Omega\varrho^2$, ϱ being the radius of gyration. Consequently: $\frac{U}{\mu} = \frac{U}{\Omega\varrho^2}$

We can call $\varrho = KU$, for ϱ is the mean of the values of u. Consequently: $\frac{U}{\mu} = \frac{U}{\Omega K^2 U^2} = \frac{1}{K^2\Omega U}$ Good employment of material consists in so arranging that $\frac{1}{K\Omega U}$ be as small as possible, or $K\Omega U$ as great.

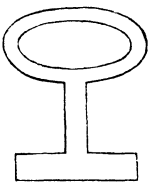
Ω is given, for we assume the material and the section of the prism. We must then so manage that K and U be great; that is to say, the surface must be extended, and the material condensed as far as possible from the axis, so that K may be as near unity as possible, or ϱ nearly equal to U.

We are led, therefore, to give the form of a double



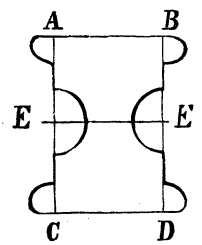
T to a prism cross section, placing thus the material at as great a distance as may be from the neutral axis—a form, in short, of flanges united by means of a plate f, which is made as thin as is consistent with solidarity. The plate f should not, of course, be too weak—so weak as to buckle or break, and solve

the connection between the two flanges. Practically, then, there must be a relation between the breadth of the plates *n* and that of the plate *f*. If a beam must at the same time resist compression as well as a cross strain, it will be necessary that the upper part, which fulfills this condition, present a form distributing the utmost possible of mass, but particularly arranged to support a pressure. Here, then, we prefer the shape

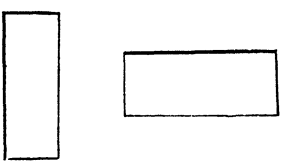


indicated in the figure. The lower part may be plane, as not being required to resist crushing efforts, but only those which tear.

If we consider any section *ABCD*, the neutral fibre *EF* bears nothing, while the extremities *AB*, *CD* support a great strain. There is then evidently an

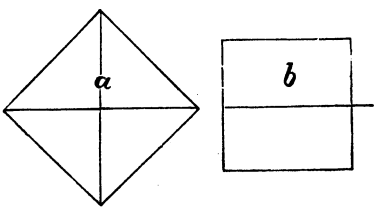


advantage in removing a certain quantity of material from *EF* to *AB* and to *CD*. But this principle cannot be applied to all substances. As for wood, economy will consist in setting a beam edgeways, and not



flat upon its side; for if the edge of cross section be but one-fourth as long as the height of the sides, then four times more beams would be necessary to offer the same resistance when set flat than when set upright.

A square piece suggests the question whether it be more suitable to place it with the sides horizontal and vertical, or with the diagonals horizontal and vertical. In the two positions, a and b, the moment



of inertia μ is the same, but V is larger in a, which admits a greater strain on the figure, and consequently is disadvantageous. To enforce the above example, we remember we found:

$$\varkappa EI = \Sigma MQ\frac{U}{\mu} \qquad \mu = A\varrho^2$$

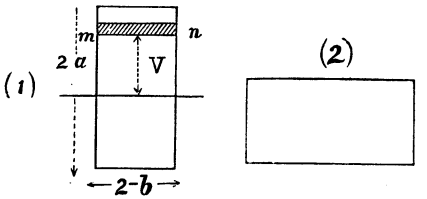
$$\varkappa EI = \Sigma MQ\frac{U}{A\varrho^2}$$

Let us call $\varrho = KU$, K being < 1. We will have then:

$$\varkappa EI = \Sigma MQ\frac{U}{AK^2U^2} = \Sigma MQ\frac{1}{AK^2U}$$

We observed that it is necessary to render this expression as small as possible; that is to say, make AK^2U maximum or K^2U maximum, consequently to make U as great as possible, which leads to increase the dimensions of the body vertically, and to make K approximate to unity, that is, to disperse the material as far as possible toward the upper and lower parts. But wood, we said, cannot be worked into special economic forms, for the material withdrawn from the centre would be lost since it could not be conveyed to the ends. In varying the placing of a

beam, however, as indicated before, let us see what is the value of $\frac{U}{\mu}$ in the two cases.



In position 1 we have, $U = a$, and considering any layer mn having a length $2b$ and a height $d\,V$, V being the distance from mn to the neutral axis, we have $u = V$. Consequently:

$$\Sigma \omega u^2 = \int 2bd\,V\,V^2 = \mu$$

$$\int 2bd\,V\,V^2 = 2b\frac{V^3}{3}$$

If we integrate from $-a$ to $+a$, we obtain:

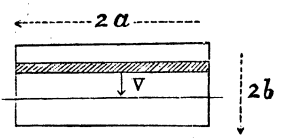
$$2b\frac{a^3}{3} + 2b\frac{a^3}{3} = \tfrac{4}{3}a^3b$$

Therefore, $U_1 = a \qquad \mu_1 = \tfrac{4}{3}\,a^3b$

$$\frac{U_1}{\mu_1} = \frac{3}{4a^2b}$$

For position 2 we shall have:

$$U_2 = b, \text{ and } \mu_2 = \int_{-b}^{b} 2ad\,V\,V^2$$



$$\mu_2 = \int_{-b}^{b} 2a\,V^2d\,V = 2a\int_{-b}^{b} V^2d\,V$$

Integrating we have: $\mu_2 = \tfrac{4}{3}b^3a$. Consequently:

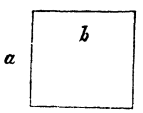
$$\frac{U_2}{\mu_2} = \frac{3}{4ab^2}$$

If we calculate the ratio of the two strains we have:

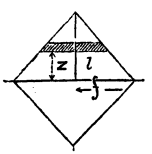
$$\frac{F_1}{F_2} = \frac{\frac{3}{4a^2b}}{\frac{3}{4ab^2}} = \frac{ab^2}{a^2b} = \frac{b}{a}$$

Whence we see, as has been remarked, that if a is four times larger than b, the strain in the position 1 will be four times smaller than in the position 2.

Again: if we consider a square section its moment

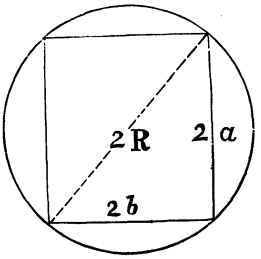


of inertia when its sides are parallel and perpendicular to the strain is $\frac{1}{12}a^4$, and that moment is the same when the diagonals also take that position, for we



have: $\Sigma wz^2 = edzz^2$. But $e = f - z$, and $\mu = fz^2dz - z^3dz = f\frac{1}{3}z^3 - \frac{1}{4}z^4$. But $f = a\sqrt{\frac{1}{2}}$, and integrating from 0 to $f = a\sqrt{\frac{1}{2}}$, we find the entire moment the same as in the first case: $\frac{1}{12}a^4$. The moments of inertia being the same the strains are as the extreme ordinates; therefore the strain on the beam in the first position is to strain in second as $\frac{1}{2}a$ is to $a\sqrt{\frac{1}{2}}$, or as $1 : \sqrt{2}$. Therefore, the strength in the two cases is as $7 : 5$.

A trunk of a tree presents generally a section of a circular form. We may ask how to cut a prism out of it so as to have the solid of greatest resistance. This is not the same thing as asking what the prism is whose volume were maximum, for we know that its section would be square. Let us designate by $2a$ and $2b$ the dimensions of the prism, such that in putting



it in position a greater dimension $2a$ may be placed vertical, according to the principle we have just enunciated and proved.

When calling $2R$ the diameter, we have $a^2+b^2 = R^2$. The strain depends on $\frac{U}{\mu}$ or $\frac{3}{4a^2b}$, and as we desire to render this expression minimum, or what is the same, make a^2b maximum, we will annul the differential coefficient of this last, which gives:

$$(1)\ a^2db+2abda = 0$$

But R being constant, $a^2+b^2 = R^2$ gives in differentiating:

$$(2)\ 2ada + 2bdb = 0$$

Subtracting after having multiplied equation (2) by b, we derive:

$$2abda+2b^2db—a^2db—2abda = 0$$

$$2b^2db—a^2db = 0$$

$$db(2b^2—a^2) = 0 \quad \text{Or, } a^2 = 2b^2 \quad \text{Or, } a = b\sqrt{2}$$

But as $a^2+b^2 = R^2$, we obtain, replacing b^2 by $\frac{a^2}{2}$:

$$a^2+\frac{a^2}{2} = R^2 \qquad 3a^2 = 2R^2$$

$$a = R\sqrt{\tfrac{2}{3}}\,;\ b = R\sqrt{\tfrac{1}{3}}$$

Such are the dimensions of the section which presents the most advantageous cutting. It is not the one, however, which gives the greatest volume of prism.

The volume being proportional to $2a2b = 4ab$, it is, in the case of a square prism, proportional to $4c^2$; c being one half the side of the square. But $4c^2 = 2R^2$. Such is the quantity proportional to the maximum

volume. Designating by v the volume of the first prism, the maximum volume being V, we have:

$$\frac{v}{V} = \frac{4ab}{2R^2} = \frac{2ab}{R^2} = \frac{2R^2\frac{\sqrt{2}}{3}}{R^2}$$

$$\frac{v}{V} = \frac{2\sqrt{2}}{3} = \sqrt{\tfrac{8}{9}}$$

SOLIDS OF EQUAL RESISTANCE.

When on the section of rupture the effort which rupture requires is not exceeded, the distribution oı material is suitable for this section; but if the solid be a uniform prism there will be an excess of material in the other sections, and we might then remove a certain quantity of matter from these, leaving only what is necessary, so that finally the entire solid may not have more or less tendency to break in one place than in another. Thus is obtained solid of equal resistance, and the considerations to be introduced for that condition are quite simple.

In any section we have:

$$\frac{E\mu}{R} = \Sigma MQ \qquad \frac{Eu}{R} = \Sigma MQ\frac{u}{\mu}$$

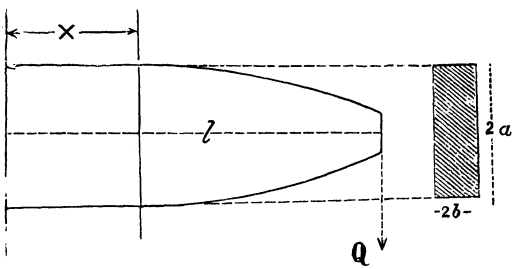
This last expression is the strain of a point the ordinate of which is u. Now in each section the strain is maximum for the point of which the ordinate is maximum. If U be this maximum ordinate, the strain in each section is represented by:

$$E\frac{U}{R} = \Sigma MQ\frac{U}{\mu}$$

If we would determine then a solid of equal resistance, that is, if we would have the resistance the same in all sections, we must pose $\Sigma MQ\frac{U}{\mu} =$ constant.

To resolve this problem we need only know the the distribution of the forces Q and notice that in passing from one section to another $\frac{U}{\mu}$ must vary in inverse ratio with the sum of the moments of the forces applied. To illustrate: let us suppose that l be a solid fastened at one end and loaded at the other with a weight Q. We have;

$$\Sigma MQ = Q(l-x)$$



Desiring now that the strain be equal over the whole length of the solid, we must state:

$$Q(l-x)\frac{U}{\mu} = \varkappa EI = \text{constant.}$$

This problem is so far undetermined, for there is an infinitude of ways of making $\frac{U}{\mu}$ vary so as to fulfill this condition.

To specify further, however: let us suppose that the section of the solid be rectangular, and that we call therefore $U = a$, and $\mu = \frac{4}{3}a^3b$, whence:

$$Q(l-x)\frac{U}{\mu} = Q(l-x)\frac{3a}{4a^3b} = \varkappa EI$$

And: $$a^2b = \frac{3}{4}\frac{Q}{\varkappa EI}(l-x)$$

This equation exhibits the law in which a and b must vary, but here, too, the problem is not determined. If, however, we assume b constant, we will find:

$$a^2 = \frac{3}{4}\frac{Q}{\varkappa EI}\frac{(l-x)}{b}$$

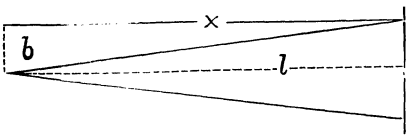
Therefore this being an equation of the form $y^2 = 2p(l-x)$, the solid of equal resistance in the case un-

der consideration is in vertical section a parabola whose vertex is at the loaded point, since $y = 0$, for $x = l$. We may also make various other suppositions upon a and b, and obtain different results.

Suppose we consider, for example, the solid as having a constant depth. We shall then write $a =$ constant, and consequently have:

$$b = \tfrac{3}{4}\frac{Q}{\varkappa EI}\frac{1}{a^2}(l-x)$$

And this, because b varies with $(l-x)$, is an equation of the first degree between b and x, that is to say, the equation of a right line, and since for $x = l$, $b = 0$, we see that the horizontal section of the



piece will be triangular. This disposition of a solid of equal resistance gives an economy of one-half, there being in its composition only half the material of the circumscribed rectangular prism.

The parabolic form similarly allows an economy of one-third.

If the prism be uniformly loaded, we shall have:

$$\Sigma MQ = p\frac{(l-x)^2}{2}$$

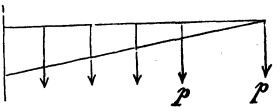
Whence: $$\varkappa EI = \frac{p}{2}(l-x)^2\frac{U}{\mu}$$

But: $$\mu = \tfrac{4}{3}a^3b$$

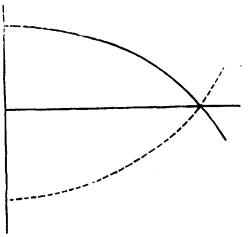
Wherefore: $$\frac{U}{\mu} = \frac{3}{4a^2b}$$

And finally: $$a^2b = \tfrac{3}{8}\frac{p}{\varkappa EI}(l-x)^2$$

If b is constant, we have here the equation of a right line, and a being 0, for $x = l$, the vertical section of the prism will be a triangle. If, however, we



assume the dimension a constant, then the equation becomes $(l-x)^2 = \frac{8}{3}\frac{\varkappa EI}{p}a^2b$, which is of the same form as the preceding equation of parabola, only the curve is turned 90°, since here $(l-x)$ represents the y, and b the x. If we then count the $(l-x)$ on the y axis, and the b on the x axis, we can construct the para-



bola. Its vertex will be on the axis of y, or in the fastened section; b is zero, for $x = l$. If it be desired to make the horizontal section symmetrical, it must be bounded by curves of two such parabolas whose ordinates are each the half of the corresponding ordinates of the above.

Suppose, finally, the case of a beam uniformly loaded and also subjected to a weight at its extremity. Then:

$$\Sigma M = \frac{p(l-x)^2}{2} + Q(l-x)$$

And:

$$\varkappa EI = \left[\frac{p(l-x)^2}{2} + Q(l-x)\right]\frac{U}{\mu}$$

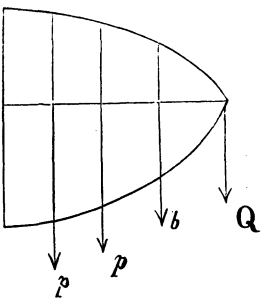
But:

$$\frac{U}{\mu} = \frac{3}{4a^2b}$$

Whence: $$a^2b = \frac{3}{8}\frac{p}{\varkappa EI}(l-x)^2 + \frac{3}{4}\frac{Q}{\varkappa EI}(l-x)$$

If we first make b constant we have the equation

between a and $(l—x)$ that of an hyperbola referred to its vertex and axes. If we make a constant we shall still have the outline of a parabolic curve, but with it axis parallel to b.



If a beam be fixed at one end and subjected to a weight at the other, with the conditions that b and a may both vary, μ being constant, we shall have: $a^3b = C$. The vertical sections of such a beam would be hyperbolas and the horizontal cubic parabolas.

For if in the equation between a, b, and $(l—x)$, $a^2b = \frac{3}{4}\frac{Q}{\varkappa EI}(l—x)$, we would observe how the depth varies with the length of the beam we need only substitute for b its value in terms of a, whereby we shall find: $\frac{C}{a} = \frac{3}{4}\frac{Q}{\varkappa EI}(l—x)$, which is the equation of a hyperbola referred to its asymtotes.

Again, substituting for a its value in terms of b, in order to determine how the breadth may vary with $(l—x)$, we find ultimately the equation $C^2b = (\frac{3}{4}\frac{Q}{\varkappa EI}(l—x))^3$, the equation of a cubic parabola.

If, instead of the condition that the moment of inertia remain constant, we impose that the area $4ab$ remain constant, we shall have, $4aba = \frac{3Q}{\varkappa EI}(l—x)$, whence $a = \frac{3Q}{\varkappa EIC}(l—x)$. This being the equation of a right line, the vertical sections of such beams would be triangles. Again substituting the value of a in

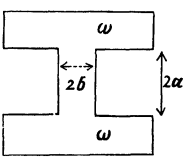
terms of b, we obtain, $\frac{C^2}{16b} = \frac{3}{4}\frac{Q}{\varkappa EI}(l-x)$, the equation of an hyperbola referred to its asymptotes, which is the curve bounding the horizontal sections.

Let us consider, for a practical illustration of a solid of equal resistance, a balance beam formed of two plates or flanges constituting the two principal parts on which we reckon for resistance, and a vertical web or plate uniting these two.

Assume the case in which the two flanges are of the same section, and let us determine the form of the balance beam that shall be a solid of equal resistance. We have:

$$U = a, \text{ and } \mu = 2\omega a^2 + \tfrac{4}{3}a^3 b$$

$2\omega a^2$ is the moment of inertia of the flanges, and

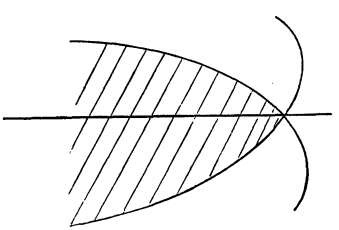


$\frac{4}{3}a^3 b$ is that of the vertical web. We shall consider 2ω as constant, and state:

$$\varkappa EI = Q(l-x)\frac{1}{2\omega a + \frac{4}{3}a^2 b}$$

$$2\omega a + \tfrac{4}{3}a^2 b = \frac{Q}{\varkappa EI}(l-x)$$

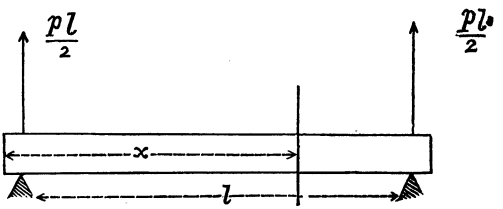
Now b is constant, and ω also; a only is variable: with $(l-x)$ the preceding equation is that of a para-



bola, for which we have $a = 0$, for $x = l$; and therefore the outline of the balance beam is composed of two arcs of parabola which cut each other at the point of application of the force.

For a balance beam the coefficient of security is assumed rather small — for example, one-third, one-fourth, or one-fifth, according to the importance of the piece—keeping always far from the limit of elasticity.

Let us now consider a body supported at its two



ends and loaded uniformly. Let p be the load supported per square metre of surface. We have:

$$\frac{E\mu}{R} = \Sigma MQ$$

$$\varkappa EI = \Sigma MQ \frac{U}{\mu}$$

$$\Sigma MQ = \frac{p(l-x)^2}{2} - \frac{p}{2} l(l-x)$$

$$= \frac{p}{2}(l^2 + x^2 - 2lx - l^2 + lx)$$

$$\Sigma MQ = \frac{p}{2}(x^2 - lx) = -\frac{p}{2} x(l-x)$$

$$\varkappa EI = -\frac{px}{2}(l-x)\frac{U}{\mu}$$

If it be supposed that all the sections are rectangular, we have:

$$\frac{U}{\mu} = \frac{3}{4a^2b}$$

Consequently:

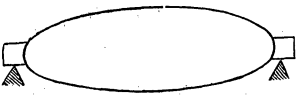
$$\varkappa EI = -px(l-x)\frac{3}{8a^2b}$$

$$a^2b = -\tfrac{3}{8}\frac{p}{\varkappa EI} x(l-x)$$

Thence we draw:

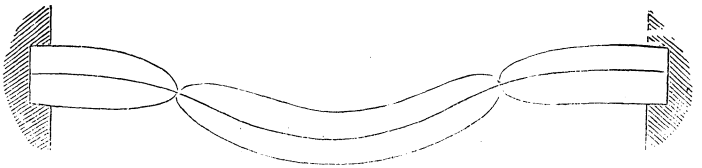
$$a^2 \frac{3p}{\varkappa EI} x^2 + \frac{3p}{8\varkappa EI} lx = 0$$

If b is constant, this equation represents an ellipse referred to its major axis and to one of its vertices. At the end we should have a section theoretically null; but evidently the solid must be reinforced in



some fashion at its ends in order to resist the shearing at those points.

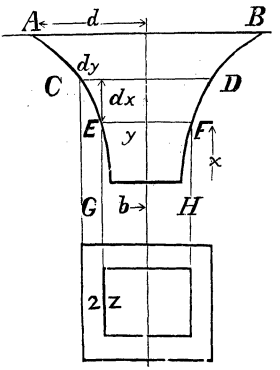
We may now again suppose a solid fastened at its two ends and loaded uniformly. Its theoretic section would be zero at the points of inflexion, since at these



points flexion is null, and consequently the strain null. Of course, this result is modified in order to keep account of the shearing strain; otherwise the beam would be outlined by two hyperbolas and an ellipse.

The form of a beam of equal resistance supporting its own weight longitudinally, may be determined as follows:

Any elementary volume of such a solid, if rectangular in cross-section, is $4yzdx$, and its weight is



$4pyzdx$; the elementary area is $4ydz + 4zdy$, and as that increment of area is supposed to support the increment of weight, we have $pyzdx = EI(ydz+zdy)$. We can always call $z = cy$, and write, therefore, $pydx=2EIdy$, or $dx = 2\frac{EI}{p}\frac{dy}{y}$, whence,

$$x = 2\frac{EI}{p} \text{ Nap. log. } y+C$$

When $y = 0$, $x = -\infty$; but if we assume $x = 0$ for some such value as $y = b$, we have,

$$0 = 2\frac{EI}{p} \text{Nap. log. } b+C$$

Whence,

$$C = -2\frac{EI}{p} \text{Nap. log. } b$$

And finally

$$x = 2\frac{EI}{p} \text{ Nap. log.} \frac{y}{b}$$

—An equation between x and y, which would serve to construct the line AG, which is the longitudinal section of the beam. If the cross section were a curve of radius y, the equation between x and y would be precisely the same. The axis of the beam is evidently an asymptote to the curve. If now we would determine how the cross section varies with x, we may write calling A any such section:

$$pAdx = EIdA$$

Or.

$$dx = \frac{EI}{p}\frac{dA}{A}$$

Or,

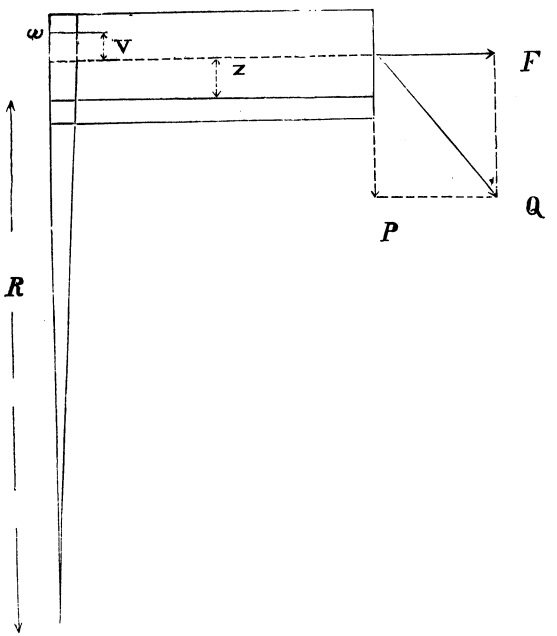
$$x = \frac{EI}{p} \text{Nap. log. } A+C$$

Or, if $A = B$ when $x = 0$,

$$x = \frac{EI}{p} \text{Nap. log. } \frac{A}{B}$$

BODIES SUBJECTED TO LONGITUDINAL AND TRANSVERSE STRAINS.

In this case there are new considerations to be introduced. Let us suppose a force Q acting upon the body, and decompose this force into two—one vertical, the other horizontal. The force F tends to elon-



gate the prism, the force P to make it bend. The upper fibres are stretched by P and by F, while the lower fibres are shortened by P and elongated by F. In result, the neutral fibre will be lower than if the force F did not exist; therefore it will be below the centre of gravity.

Assume two vertical sections adjoining. Let R be the radius of curvature corresponding to the neutral fibre, and let us consider a fibre the ordinate of which is V referred to the centre of gravity. Let z

be the ordinate of the centre of gravity referred to the neutral axis, the elongation of the fibre is proportional to $\frac{V+z}{R}$, and the strain for a section ω infinitely small, situated at the distance $V+z$ from neutral fibre is : $\varphi = E\frac{V+z}{R}\omega$

$$\Sigma\varphi = \frac{E}{R}\Sigma[\omega(V+z)] = \frac{E}{R}[\Sigma\omega V + \Sigma\omega z]$$

But $\Sigma\omega V = 0$, since it is the sum of the moments of the elementary areas referred to the centre of gravity. Therefore, $\Sigma\varphi = \frac{E}{R}\Sigma\omega z = E\Omega\frac{z}{R}$. But we must also have for equilibrium, $\Sigma\varphi + F = 0$, whence $-F = E\Omega\frac{z}{R}$; a formula which may serve to determine z when we know R or $\frac{\mu}{\varkappa EI}$. A transverse resistance is of course developed, which equilibrates P, and moreover as before the sum of the moments of φ in relation to the centre of gravity, since the moment of $F = 0$ must equal the sum of the moments of the forces P.

Instead of the sum of the moments of P we may take the moments of the forces Q, since the moment of F is null. We shall have then :

$$\Sigma MQ = \Sigma MP = \Sigma M\varphi$$

$$\Sigma MQ = \frac{E}{R}\Sigma\omega(V+z)V$$

$$\Sigma MQ = \frac{E}{R}(\Sigma\omega V^2 + \Sigma\omega Vz)$$

Now $\Sigma\omega V = 0$; therefore $z\Sigma\omega V = 0$; consequently :

$$\Sigma MQ = \frac{E}{R}\Sigma\omega V^2 = \frac{E\mu}{R}$$

This is the same formula for moment of flexure as in the preceding cases, remarking however, that R is the radius of curvature of the neutral fibre, which here is not really the locus of the center of gravity, yet in most applications we may consider $\left(\frac{1}{\frac{d^2y}{dx^2}}\right)$

as representing the radius of curvature of the neutral fibre because it differs little from that of the centres of gravity, in fact, calling R' the radius of the fibre passing through the centre of gravity, we have:

$$R' = R+z = R-\frac{FR}{E\Omega}$$

consequently:

$$R' = R\left(1-\frac{F}{E\Omega}\right)$$

Now: $\frac{F}{E\Omega}$ being evidently excessively small, R' differs very little from R and we may therefore call $R' = R$ for the approximations within which any theory is valid.

The strain at a given point will be

$$\varkappa EI = \Sigma MQ\,\frac{V+z}{\mu},$$

or,

$$\varkappa EI = \frac{\Sigma MQV}{\mu}+\frac{\Sigma MQz}{\mu};$$

but

$$\frac{\Sigma MQz}{\mu} = -\frac{F}{\omega}$$

Hence,

$$\varkappa EI = \frac{\Sigma MQV}{\mu}-\frac{F}{\omega}$$

Therefore in calculating the dimensions of a body subjected to a cross strain, and a longitudinal one at the same time, we must first examine separately for the longitudinal strain it will bear, or for $\varkappa EI = \frac{F}{\omega}$ and then for the cross strain $\varkappa EI = \frac{\Sigma MQ}{\mu}-\frac{F}{\omega}$ at the point of maximum strain. The greater of these strains must determine the dimensions of the piece. It may happen, however, that a longitudinal strain of compression may increase flexion, and that its moment therefore should be added to the moment of the applied transverse force.

Let us first then take up the case of a vertical

prism or column fastened at the foot and loaded at the top, which is free.

Let us suppose that on account of the vibrations of any structure or for whatever reason the prism be bent a certain quantity, and let us seek its position of equilibrium: We have then relative to the centre of gravity

$$E\mu \frac{d^2y}{dx^2} = \Sigma MQ$$

Wherefore:

$$E\mu\frac{d^2y}{dx^2} = F(f-y)$$

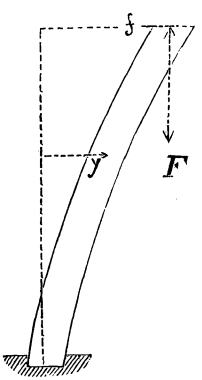
$$E\mu\frac{d^2y}{dx^2}dy = F(f-y)dy$$

Integrating, we have:

$$\tfrac{1}{2}E\mu\left(\frac{dy}{dx}\right)^2 = F\left(fy-\frac{y^2}{2}\right)+\text{constant.}$$

For $y = 0$ we have $x = 0$; consequently $C = 0$: therefore:

$$\left(\frac{dy}{dx}\right)^2 = \frac{F}{E\mu}(2fy-y^2)$$



And:

$$dx\sqrt{\frac{F}{E\mu}} = \frac{dy}{\sqrt{2fy-y^2}} = \frac{\frac{dy}{f}}{\sqrt{1-\left(1-\frac{y}{f}\right)^2}}$$

Integrating, we find :

$$x\sqrt{\frac{F}{E\mu}} = \text{arc cos.}\left(1-\frac{y}{f}\right)+C$$

From this equation, since $C = 0$, because when $y = 0$, x also equals 0, we derive :

$$1-\frac{y}{f} = \cos.\ x\sqrt{\frac{F}{E\mu}}$$

$$y = f\left(1-\cos.\ x\sqrt{\frac{F}{E\mu}}\right)$$

We cannot make F arbitrary, for when $x = l$ we have $y = f$, and consequently :

$$l\sqrt{\frac{F}{E\mu}} = (2n+1)\frac{\pi}{2}$$

Or,

$$l^2\frac{F}{E\mu} = (2n+1)^2\frac{\pi^2}{4}$$

Whence,

$$F = \frac{(2n+1)^2\pi^2 E\mu}{4l^2}$$

And replacing F, we have :

$$y=f\left(1-\cos.\ x\sqrt{\frac{(2n+1)^2\pi^2}{4l^2}}\right)$$

$$y=f\left(1-\cos.\ x\frac{(2n+1)\pi}{2l}\right)$$

By giving n different values, we derive various corresponding ones for y, and for F.

$n = 0 \qquad F_0 = \frac{E\mu}{l^2}\frac{\pi^2}{4} \qquad y = f\left(1-\cos.\frac{x}{l}\frac{\pi}{2}\right)$

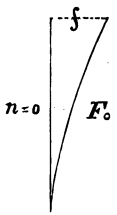
$n = 1 \qquad F_1 = 9F_0 \qquad y = f\left(1-\cos.\frac{x}{l}\frac{3\pi}{2}\right)$

$n = 2 \qquad F_2 = 25F_0 \qquad y = f\left(1-\cos.\frac{x}{l}\frac{5\pi}{2}\right)$

$n = 3 \qquad F_3 = 49F_0 \qquad y = f\left(1-\cos.\frac{x}{l}\frac{7\pi}{2}\right)$&c.

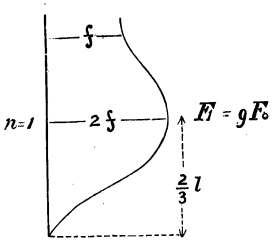
Let us discuss $n = 0$. We have the value of a fraction $\frac{x}{l}$ of $\frac{\pi}{2}$, increasing from 0 to 1, whose cosines will vary, therefore, from 1 to 0, while y increases regularly from 0 to f. If $n = 1$, the fraction in-

creases regularly from 0 to 3, for $x = \frac{l}{3}$ we have $y = f$,



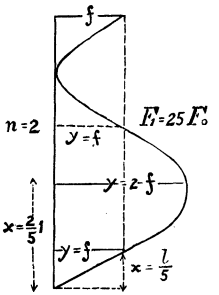
and for $x = \frac{2l}{3}$ we have $y = 2f$. For $n = 2$ we reason in the same manner, upon the values of the variables ; as, for $x = \frac{l}{5}$, $y = f$, for $x = \frac{2l}{5}$, $y = 2f$, &c., &c., and so on.

In proportion as n increases, the crushing force increases, varying with the square of the value of u.

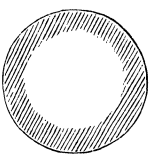


The effort F will be the greater as the prism has less length ; and greater moment of inertia its resistance is in inverse ratio to the square of its length. $F = \frac{E\mu}{l^2} \frac{\pi}{4}$ is the least force that can make the prism bend, and the force that will produce crushing is $P = AEI$. Therefore, as F may be less or greater than P, will the prism be in danger of breaking by crushing or by flexion. In order that it resist compression suitably, the section must be increased and the length diminished ; but to resist an effort of flexion its moment of inertia is most important.

If a prism have the same amount of inertia in reference to any axis whatever in its cross section, (square, circle,) the body evidently does not tend to



bend to one side rather than to another. If, however, a column have an unsymmetrical section, it will tend to bend in a plane perpendicular to the major axis of the figure, because in this direction the moment of inertia is least. As the material, also, should be condensed far from the centre of gravity, as well as



symmetrically, therefore the form of a hollow column of circular section will be the most suitable for bodies in the above condition. The radius must be large and the thickness small.

If we consider the value $F = \frac{(2n+1)^2\pi^2 E\mu}{4l^2}$ and assume the column to be square in section, we shall have: $F = \frac{E(2n+1)^2\pi^2}{48}\frac{c^4}{l^2}$; c being the side of the square, whence:

$$c^2 = \frac{4l}{(2n+1)\pi}\sqrt{\frac{3F}{E}}$$

which shows that the section must be proportional to the length to support a given weight. The same may be made apparent when considering the section as circular, since we shall find:

$$\pi r^2 = \frac{2l}{2n+1}\sqrt{\frac{2F}{E\pi}}$$

If we now consider a prism like the above, but so placed that the upper end shall remain in the same vertical, it will be sufficient in the preceding equation to make $f = 0$, which gives:

$$E\mu\frac{d^2y}{dx^2} = -Fy \qquad \frac{E\mu\, d^2y}{dx^2}dy = -Fydy$$

$$E\mu\left(\frac{dy}{dx}\right)^2 = -Fy^2 + C$$

Now $y = f_1$ ought to give $\frac{dy}{dx} = 0$: consequently $C = Ff_1^2$, and $E\mu\left(\frac{dy}{dx}\right)^2 = F(f_1^2 - y^2)$; whence,

$$dx\sqrt{\frac{F}{E\mu}} = \frac{dy}{\sqrt{f_1^2 - y^2}}, \text{ and}$$

$$x\sqrt{\frac{F}{E\mu}} = \text{arc sin.}\left(\frac{y}{f_1}\right) + C$$

For $x = 0$, we have $y = 0$, therefore the constant is null, and consequently:

$$y = f_1 \text{ sin. } x\sqrt{\frac{F}{E\mu}}$$

If we make $x = l$, the sine of $l\sqrt{\frac{F}{E\mu}}$ must be zero, because $y = o$, and we shall have:

$$l\sqrt{\frac{F}{E\mu}} = n\pi, \text{ or } \sqrt{\frac{F}{E\mu}} = \frac{n\pi}{l}$$

and consequently, by replacing this last in the value of y,

$$y = f_1 \text{ sin. } \frac{nx}{l}\pi$$

We have also:

$$F = \frac{E\mu}{l^2}n^2\pi^2$$

We need not consider the case of $n = 0$, in which $F = 0$.

Let

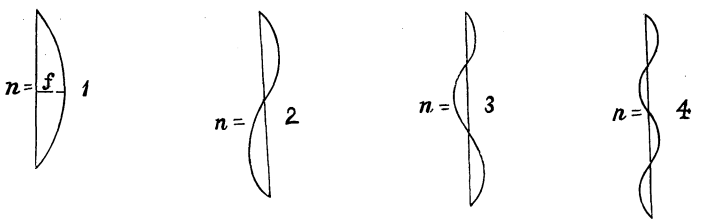
$$n = 1, \text{ then } F = \frac{E\mu}{l^2}\pi^2$$

This force F is four times greater than that found in corresponding case when the upper part of the prism was movable. Thus then by establishing the prism without necessarily fastening it, but arranging

so that the upper part shall remain in the same vertical with the lower, the minimum force which will produce flexion will be $E\frac{\mu}{l^2}\pi^2$, instead of $\frac{1}{4}\ E\frac{\mu}{l^2}\pi^2$

These forces, when n varies, as 1 : 2 : 3 : 4, vary themselves as 1 : 4 : 9 : 16. The different forms of the axis are thus :

$F = 1.$ $F = 4.$ $F = 9.$ $F = 16.$

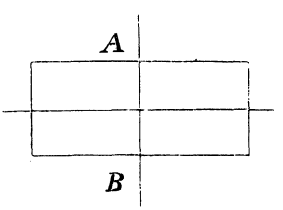


If having then a series of prisms we attach them together at their middle points so as to prevent them there from bending, we quadruple the force which they can support, for that amounts to making $n = 2$. If the pieces are braced at one-third and two-thirds their length, their resistance is rendered nine times greater, &c. Such is the effect of bracing.

When any piece is not braced, we have :

$$F = \frac{E\mu}{l^2}\pi^2 = E\pi^2\frac{\mu}{l^2}.$$

This resistance depends upon the moment of inertia relative to an axis passing through the centre of gravity of the section. If the section be a rectangle,



the flexion will tend to occur in the plane passing through $A\ B$, and again, if as in the preceding case, we consider the force F' which might occasion crushing, we have:

$$F' = EIA.$$

F' is therefore proportional to A, whilst F is independent of it, varying only with the moment of inertia.

In considering the two forces F and F', practically, at any time, it is necessary to observe which is the greater of the two, and consequently the one for which the column must be measured.

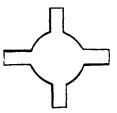
We have,

$$\frac{F'}{F} = \frac{1}{\pi^2} A \frac{l^2}{\mu} = \frac{1}{\pi^2} \frac{l^2}{\varrho^2}, \text{ for } \mu = A\varrho^2.$$

$\frac{1}{\pi^2}$ is a known quantity; wherefore, for a given material, a prism will better resist flexure or crushing according to the variation of $\frac{l^2}{\varrho^2}$. If l be great and ϱ small, we shall have: $F' > F$, and consequently, the force necessary to produce crushing greater than that necessary to produce flexion, or flexion will take place before crushing.

If in this prism we could consider the moment of inertia only in reference to one axis, we might assume the form of double T for its cross section, but we must, in fact, oppose flexion in every direction; wherefore, the hollow column of circular form is preferred.

If the material cannot be hollow (wood), the column



may be shaped with a full core, reinforced by symmetrical flanges.

If the column being short, need only resist a crushing force, it will be made solid always. If in this case also, we reconsider the value of F and

assume the section to be either square or circular, we shall find in the first supposition :

$$c^2 = \frac{2l}{n\pi}\sqrt{\frac{3F}{E}}$$

And in the second:

$$\pi r^2 = \frac{l}{n}\sqrt{\frac{2F}{\pi E}}$$

These formulas do not consider the weight of the column. The following are empiric formulas proposed by various engineers :

First, for wooden posts : $\frac{P'}{c^2} = 2565\left(\frac{c}{l}\right)^2$ a formula of Hodgkinson, in which P is in kil., c in centimetres and l in decimetres—this is moreover for rupture. The coefficient of security to be applied to such is $\frac{1}{20}$ to $\frac{1}{7}$. Morin gives $P = A\frac{c^4}{l^2}$ in which P is in kil., c in centimetres, and l in decimetres ; and for A, he gives the following values :

Oak, - -	A = 256.5 to 180.
Cedar, - - - -	214.2.
White and Yellow Pine,	160.

These values of A give P a safe load without using a further coefficient. A formula of Hodgkinson, for cast iron pillars, is $P = 10676\frac{d^{3\cdot6}}{l^{1\cdot7}}$; the units being the same as above.

This is easily calculated by logarithms ;

$$\log. d = \frac{\log. P + 1\cdot7 \log. l - \log. 10676}{3\cdot6}$$

and represents a safe load.

Mr. Love gives for cast iron :

$$P = \frac{1250d^4}{1\cdot846\, d^2 + 0\cdot000\,43l^2}$$

the unit being kil. ; centimetres ; and decimetres.

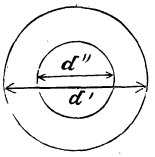
For forged iron columns, the same engineer gives

$$P = \frac{600d^4}{1\cdot973\, d^2 + 0\cdot000\,64l^2}$$

In both these the load P is within safety limits.

It is admitted that the resistanee of hollow columns

is equal to the difference of resistance of two solid ones of diameters d' and d''; but that practically for



a good casting, $d'—d'' = e$ must bear a certain relation to l. For instance, when

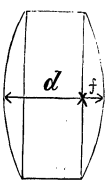
$$l = 2 \text{ to } 3 \text{ met. } e \geqq 12 \text{ mill.};$$

$$l = 3 \text{ to } 4'' ; e \geqq 15 \text{ mill.};$$

$$l = 4 \text{ to } 6 \text{ met. } e \geqq 20 \text{ mill.};$$

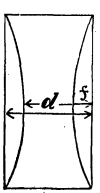
$$l = 6 \text{ to } 8 \text{ met. } e \geqq 25 \text{ mill.}$$

The longitudinal compression of columns is said to be accompanied by a bulging, maximum at the middle



section, in which the proportional increase of the diameter is one-fourth the proportional compression of the column.

The converse also is stated of prisms submitted to extension—the middle diameter being diminished in



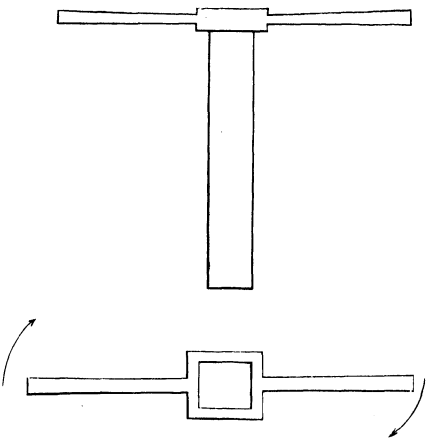
the same proportion, whence:

$$\frac{f}{d} = \frac{1}{4}\,\frac{\lambda}{l}$$

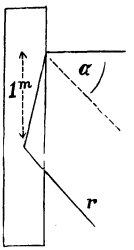
This variation is evidently almost insensible.

RESISTANCE TO TORSION.

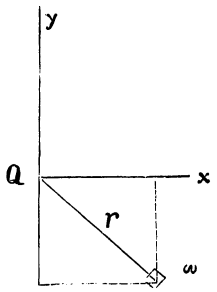
Consider a vertical axle fastened at one end and subjected to a couple tending to twist it without



bending. The amount of torsion is measured by the angle formed after torsion by two radii, that were primitively in the same vertical plane, passing through the neutral axis, and situated at a metre's distance, the one above the other.



If in any section whatever we consider an element ω, situated at a distance $\sqrt{x^2 + y^2}$ from the neutral fibre, it must have developed a resistance equal to the torsion in a direction inverse to that force, and all the resistances developed above the section considered, give a sum of moments equal to the moment



of the couple. Any of these forces is proportional to the element ω, to the angle of torsion and to the radius r. We have then for its expression: $a\omega rt$, t, being a numerical coefficient to be determined by experiment. This force is perpendicular to the radius r, for it acts in the direction in which the molecules tend to separate.

Decomposing it parallel to two axes in the section, we obtain, parallel to the axis of y:

$$a\omega rt\frac{x}{r} = a\omega tx$$

Parallel to the axis of x:

$$a\omega rt\frac{y}{r} = a\omega ty$$

Moreover,

$$\Sigma a\omega tx = o, \text{ and } \Sigma a\omega ty = 0$$

Or, since t is a constant,

$$\Sigma a\omega x = 0, \text{ and } \Sigma a\omega y = 0$$

Moreover,

$$\Sigma \omega x = 0, \text{ and } \Sigma \omega y = 0.$$

Now if $\Sigma \omega x = 0$ and $\Sigma \omega y = 0$, it shows that the point through which the neutral axis passes is the centre of gravity of the seetion, because this property holds only for the centre of gravity.

If we write the sum of the moments of the forces relative to the point 0, we shall have:

$$\Sigma a\omega rtr = at\Sigma \omega r^2 = at\mu$$

And, also,

$$at\mu = M$$

M being the moment of the forces that tend to produce the torsion of the prism.

Having the moment of torsion, we may now demand the strain. The force being $at\omega r$, the strain referred to the unit of surface is atr, or $\frac{M}{\mu}r$: then, $\frac{Mr}{\mu}$ is the strain supported by a point situated at the distance r. This is the quantity which must not exceed a given limit, and which must be diminished as far as possible for a given surface in order that the body resist to advantage.

In the quantity $\frac{r}{\mu}$, μ is not susceptible of varying in a given section, for we have $\mu = A\varrho^2$, and $\frac{r}{\mu} = \frac{r}{A\varrho^2}$ but ϱ is the mean radius, which multiplied by A, gives μ. We may therefore call $\varrho = Kr$, and, consequently, write $\frac{r}{\mu} = \frac{r}{AK^2r^2} = \frac{1}{A}\frac{1}{K^2r}$, an expression independent of μ.

In order that the fatigue be as light as possible, K and r must be increased. K is smaller than unity, but may have a value as near unity as we please, and therefore to obtain the maximum resistance of a section, we must arrange the entirety of its molecules at as great a distance as possible from its centre of gravity, which leads to the form of a hollow cylinder, as thin as practicable.

When a hollow form can not be constructed, we may yet be able to assume a square, or a circular one, or else a form of flanged section. It may be asked whether to resist torsion the square or the circular section be best for equal full areas; to answer which, it is necessary to comparatively estimate μ in the two cases. This is the moment of inertia relative to the centre of gravity of the section, or,

$$\mu = \Sigma\omega r^2$$

Let us consider then first, a circular section, and in it an annulus situated at a distance S from the the centre and having a breadth ds. We shall have:

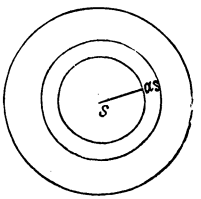
$$\Sigma\omega r^2 = \int_0^R 2\pi S\, dS \times S^2$$

Seeing that

$$\omega = 2\pi s ds$$

Therefore:

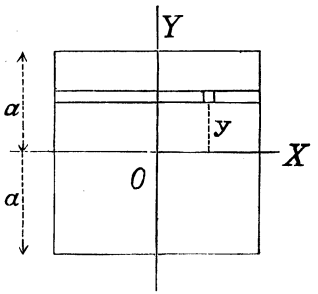
$$\Sigma\omega r^2 = \int_0^R 2\pi S^3\, ds = 2\pi \frac{S^4}{4} = \pi\frac{S^4}{2} \text{ or } \mu = \pi R^2 \frac{R^2}{2}$$



If we represent by A the section of the circle, we have:

$$A = \pi R^2, \text{ consequently: } \mu = A \frac{R^2}{2}$$

Let us now consider the square, and estimate $\Sigma\omega r^2$ for it. If we assume two symmetrical rectangular axes X and Y, the moment of inertia of the square



relative to its centre is equal, we know, to the sum of the moments of inertia relative to the two axes. Now the moments relatively to the two axes are evidently equal, on account of the symmetry, and we may then state:

$$\Sigma\omega r^2 = 2\, \Sigma\omega y^2,$$

but,

$$\Sigma\omega y^2 = \int_{-a}^{a} 2a\, dy\, y^2 = 2a \int_{-a}^{a} y^2\, dy \cdot$$

Now,

$$\int_{-a}^{a} y^2\, dy = 2\,\frac{a^3}{3}$$

consequently,

$$\Sigma\omega y^2 = 2a\, 2\,\frac{a^3}{3} = \tfrac{4}{3}\, a^4$$

and,

$$\Sigma\omega r^2 = \tfrac{8}{3}\, a^4 = 4a^2\,\frac{2a^2}{3}$$

but,

$$4a^2 = A, \text{ therefore: } \mu = A\,\frac{2a^2}{3}$$

We find then for the circle:

$$\mu = A\,\frac{R^2}{2}. \text{ For the square: } \mu = A\,\frac{2a^2}{3}$$

But, we have:

$$\alpha = \frac{M}{t}\,\frac{1}{\mu}$$

and if we apply, there will result for the circle:

$$\alpha = \frac{M}{t}\,\frac{2}{AR^2}$$

for the square:

$$\alpha' = \frac{M}{t}\,\frac{3}{2Aa^2}$$

or,

$$\frac{\alpha}{\alpha'} = \frac{3\ AR^2}{2\times 2\ A\ a^2} = \tfrac{3}{4}\,\frac{R^2}{a^2}$$

but,

$$4a^2 = \pi R^2$$

since the surface of the circle is equal to that of the square, wherefore:

$$\frac{\alpha}{\alpha'} = \frac{3R^2}{\pi R^2} = \frac{3}{\pi} = \frac{3}{3.1416}$$

We see then that:

$$\frac{\alpha}{\alpha'} < 1 \text{ or } \alpha < \alpha'$$

Let us now consider the strain. For the circle we have:

$$\alpha t r = M\,\frac{r}{\mu} = M\,\frac{2}{AR} = f$$

For the square:

$$atr = M\frac{r}{\mu} = M\frac{3}{Aa}\sqrt{2} = f^1$$

Which of the two strains is the greater? We have:

$$\frac{f}{f^1} = \frac{2}{R}\,\frac{\sqrt{2}\,a}{3} = \frac{2\sqrt{2}}{3}\,\frac{a}{R}$$

Now:

$$\frac{a}{R} = \frac{\sqrt{\pi}}{2}$$

since $4a^2 = \pi R^2$, then:

$$\frac{f}{f^1} = \frac{2\sqrt{2}}{3}\,\frac{\sqrt{\pi}}{2} = \frac{\sqrt{2\pi}}{3} < 1,$$

or,

$$f < f^1$$

The strain is then less in a circular than in a square axle. The angle of torsion is greater for the circular, but the strain is less; because in the square there are points which strain extremely. These are the ends of the diagonals.

SHEARING.

When a beam is subjected to a transverse strain, one of the conditions of equilibrium in any section has been stated under the form $\Sigma Q = \Sigma Q'$; that is to say, there are developed forces Q' equal and opposed to Q in the section considered. These forces Q', have no moments with regard to any axis in the section, and represent the resistance of the beam to shearing. The coefficient of resistance to transverse shearing, as in case of rivets, &c., is assumed the same as that to extension, and a cross section fitted to resist a certain shearing force is therefore calculated precisely as if that force were one of extension. The tendency to shear at any section in given cases is found by determining ΣQ for such section. Thus, in case of a beam fixed at one end and subjected to ΣQ at the other, the shearing stress at any and every

section is ΣQ. If the beam be loaded at the middle point by a weight Q, the shearing stress is constant in every section and equal to $\frac{1}{2}Q$. In case the beam be uniformly loaded, the shearing stress at any point is $\Sigma Q = \frac{1}{2}pl - px$, which is maximum for $x = 0$, and becomes null for $x = \frac{1}{2}l$. In solids of equal resistance at the points where flexure is null, the cross section must still be calculated in proportion to ΣQ at those points.

Since also under the action of a cross strain the fibres of a beam are unequally elongated and shortened, there must be a stress of longitudinal shearing of the fibres upon one another, and this developes a sort of supplementary resistance to flexure in the beam, since not only each fibre resists compression or extension, but is also reinforced in that resistance by its contiguous fibres. The resistance to this sort of shearing would not naturally be thought to be equal to the resistance of the fibre itself to compression or extension, though perhaps somewhere near it. If φ represent this resistance, and T and C the resistances to extension and compression, Barlow finds for wrought iron, $\varphi = 0{\cdot}53\ T$, and $0{\cdot}8\ C$ nearly.

This resistance and its moment might be introduced into the formulas for determining the strength of beams, and Barlow does so, but he considers that these forces are of uniform intensity at all points of a section, that is to say, not varying with $\frac{U}{\varrho}$ like the similar forces $\varphi\ \varphi'$ which represented the resistance of the fibre itself. There seems no reason for this, and if EI be assumed from experience of different material in the circumstances to which the common formulas are intended to apply, they will be sufficient to safely determine all the variations of dimensions desirable.

There must also evidently exist during torsion a certain resistance to torsional shearing, which increases the resistance of an axle to the spiral elongation of its fibres.

NUMERICAL COEFFICIENTS.

Forged Iron:

$$E = \begin{cases} 16{,}000{,}000{,}000 \\ 18{,}000{,}000{,}000 \\ 20{,}000{,}000{,}000 \end{cases}$$

Limit of elasticity, 15 kg. per sq. millimetre.

$$P = EI = 15{,}000{,}000 \text{ kg.}$$
$$I = \frac{15}{20{,}000} = 0{,}000\,75$$

Rupturing load: 25 kg. for compression.
36 kg. for extension.

These numbers vary with the quality of the iron, and apply to the large iron employed in machines. In fine pieces which have undergone more careful elaborations, the resistance increases.

A coefficient of security depending on the importance of the piece, must always be introduced; and generally not desiring to exceed half the limit of elasticity, we will not load more than 7 kg. per millimetre square, or in short, keep within 6 kg. For important pieces, it is best to restrict even within this. Watt in his engines made the diameter of the piston-rod $\frac{1}{10}$th the diameter of the piston. The engine being low pressure, the load on the rod was about 1 kilogramme per millimetre square, therefore he thus assured a complete security.

For a suspended bridge we may keep between 2 kg. and $2\frac{1}{2}$ kg. per mm. sq., or if it be a wire bridge, which resists better, we may load from 4 to 5 kg. per sq. mil.

The resistance of iron is maximum towards 100°; at 300° it is the same as when cold. In very cold weather resistance is less than at the ordinary temperature.

The resistance to rupture by shearing is considered as equal to the resistance to extension; *i. e.*, to 36 kg. per square millimetre.

STEEL.—We admit that some of the coefficients are the same as or a little greater than those of iron.

$$E = 20{,}000{,}000{,}000 \text{ kg.}$$

The limit I is extended however, and the resistance to rupture is found to be 75 kg. per square millimetre instead of 36.

In applications we must note indeed the quality of steel. The employment of a certain steel supposes that experiments have been made indicating the numbers that may be used for coefficients. We admit, however, that a force which yet does not injure its elasticity may be three times greater than that for iron, or the elastic resistance of steel is three times that of iron. The employment of steel will then permit a diminution of the thickness of pieces.

CAST IRON.—For castings we have:

$$E \text{ comprised between } \begin{cases} 9{,}000{,}000{,}000 \\ 12{,}000{,}000{,}000 \end{cases}$$

Resistance to rupture is comprehended between 11, 13 and 18 kilo. per square millimetre; we have, however:

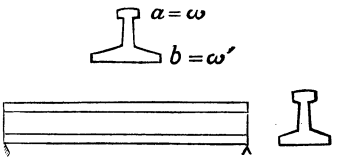
$$F = 11{,}000{,}000 \text{ for extension,}$$

but,

$$F = 70{,}000{,}000 \text{ for compression.}$$

Castings therefore resist compression better than extension, and resist crushing better than wrought iron, which resists extension better. Instead of giving beams of cast iron, the form of symmetrical double T, which is the most suitable when the resistance to extension is the same as to compression we must use the form modified, so that the resistance may be the same throughout. Though it is not certain that this would be necessary were the cross

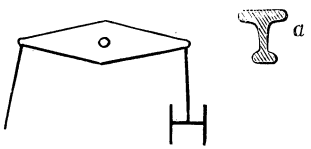
section large enough to retain the strain quite within the elastic limit.



For an extension strain, we may safely load 2 K. to 2.25 K. per mm. sq., and 10 to 12 kg. for crushing. If we consider then the two parts a and b, we must compose then so as to bear a load of 2 kg. per millimetre square in the upper member, and 10 kg. in the lower. If, therefore ω and ω' be the sections at a and b, we ought, evidently, to make $10\omega = 2\omega'$, whence:

$$\frac{\omega}{\omega'} = \frac{2}{10} \quad \omega = \tfrac{1}{5}\,\omega'$$

If instead of a fixed beam, we consider the case of a balance-beam, we must first distinguish between beams of single and double acting engines. When the engine is single acting, its balance-beam ought to have the section (a), because the top flange is



always strained by extension. When the engine, however, is double acting, then the two flanges work alternately by extension and by compression, and they should, therefore, be equal. Generally they are so in single acting engines, but the strain is greater for the upper flange.

Copper.—It is generally admitted that the resistance of sheet copper is $\frac{2}{3}$ that of sheet iron. Sheet copper is less resisting in proportion as its temperature is higher. Designating its resistance by 1 at the ordinary temperature, it is 0.90 at 150°, 0.50 at 400°, and 0.33 at 500°.

WOOD.—Oak, Yellow Pine:

$$E = 980{,}000—1{,}200{,}000.$$

White Pine:

$$E = 569{,}000.$$

Limit of elasticity:

Oak,	$2^{kg}.85$	per mm.	square.
Yellow Pine,	$2^{kg}.15$	"	"
White Pine,	$1^{kg}.63$	"	"

Limit of rupture:

Oak,	$5^{kg}.66$	per mm.	square.
Yellow Pine,	$4^{kg}.18$	"	"
White Pine,	$2^{kg}.48$	"	"

Ordinary loads:

$0^{kg}.6$ to $0^{kg}.8$ when there is extension.
$0^{kg}.4$ to $0^{kg}.5$ when compression.

In choice timbers these numbers may be noteably exceeded, and when the pieces are long, sometimes as far as the half of the limit of elasticity, *i. e.*, assuming $\alpha = \frac{1}{2}$. We have to do so when the pieces require great dimensions.

For torsion, few exact data are given, but one may be allowed to assume for

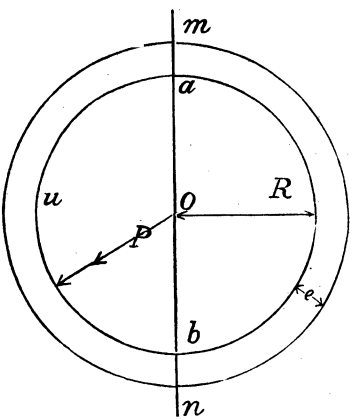
Wrought Iron and Steel,	$t = 6{,}000{,}000{,}000$
Cast Steel, - -	12,000,000,000
Cast Iron, - - -	2,000,000,000
Oak, Pine, - -	400,000,000

To obtain t as above, an angle of torsion is determined, and consequently the elongation proportional to the section; α is generally taken, $\alpha = \frac{1}{10}$ where rupture is in question.

The data on torsion are less exact than on compression and extension; and also vary considerably with the quality of materials.

CALCULATION OF THE THICKNESS TO BE ALLOWED FOR A BOILER.

Let P' be the internal pressure per square millimetre, and let us consider a diametral plane in which rupture tends to be effected, and examine what forces are in play to produce this rupture.



The forces tending to open the cylinder at m are the horizontal components of the pressure P' upon the half circumference multiplied by a certain length h of the boiler. This pressure will be therefore:

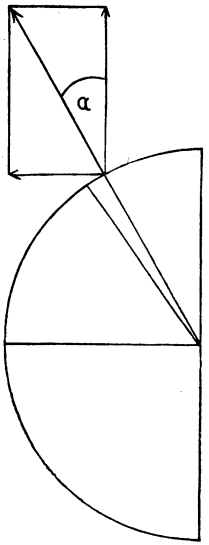
$$P'2Rh$$

This is easily seen because the pressure on the plane aob is the same as the horizontal component pressure on the half circumferences aub.

For $P'Rd\alpha$ = normal pressure on $Rd\alpha$ and $P'Rd\alpha$ sin. α = horizontal component.

Whence,

$$\int_0^\pi P'Rd\alpha \text{ sin. } \alpha = \text{whole pressure.}$$



But,

$$\int_0^{\pi} P'Rda \sin. a = -(\cos. \pi - \cos. 0)\ P'R = 2P'R$$

The external pressure is:

$$Ph2(R+e)$$

The difference of pressure tending to produce rupture is:

$$2P'Rh - Ph2(R+e) = (P'-P)\ 2Rh - 2Phe$$

The section opposed to rupture is $2eh$, and it offers then per unit of surface an effort S such that we have:

$$S2eh = (P'-P)\ 2Rh - 2Phe$$

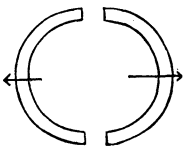
and,

$$Se = (P'-P)\ R - Pe$$

but P' being very small relatively to S, for P is the atmospheric pressure per square metre, and less than 0.001 of S, we may write, instead of $(S-P)e = (P'-P)R$,

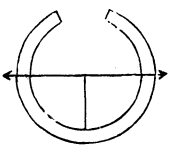
$$Se = (P'-P)R$$

Then $P'—P$ being given as well as R and S, we may determine the thickness e. This is proportional to the radius of the boiler and to the excess of the internal over the external pressure, and its value is:



$$e = \frac{P'—P}{S} R$$

Again, the cylinder might naturally enough open along a single generatrix instead of bursting apart, but the equation would not be changed. The force that tends to produce rupture neglecting e beside R would be:



$$(P'—P)h\ 2R$$

Its lever arm is R, consequently the moment relative to the point around which rupture tends to occur is:

$$(P'—P)hR2R$$

This moment must equal the moment of the force which is developed at the upper part to resist rupture. This force is Seh, wherefore we may write:

$$2(P'—P)h\ R^2 = Seh2R, \text{ or, } (P'—P)R = Se$$

The practical formula in handbooks is:

$$e' = 0^m0018d(n—1) + 0^m003$$

If we consider this formula limited to the first term in the second member, we have:

$$e' = 0^m0018\ d(n-1)$$

In this formula we see also that the thickness is made proportional to the diameter and to the excess of the internal pressure over the external pressure. The constant term added, aims to supply, for wear and tear, an excess of resistance; it is moreover necessary to append such a term, since for low pressure engines in which $P' = P$ we should be led to give the boiler otherwise a thickness $= 0$.

Let us compare this practical value of e' with the theoretical e. Equalizing the two, we obtain

$$0.0018D\ (n-1) = \tfrac{1}{2}\ \frac{P'-P}{S} D$$

whence,

$$S = \tfrac{1}{2}\ P'-P\ \frac{1}{0.0018\ (n-1)}$$

P is 10334 kg.

P' is $n10334$

$P'-P$ is then $(n-1)\ 10334$

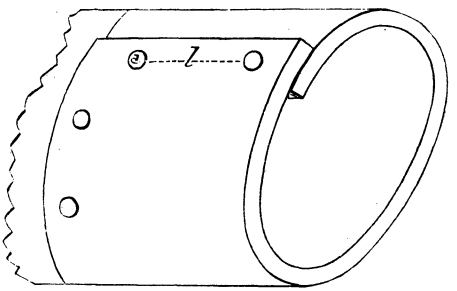
$$S = \tfrac{1}{2}\ \frac{10334\ \text{kg.}}{0.0018} = \frac{10334\ \text{kg.}}{0.0036}$$

Thus in calculating the thickness of a boiler by the formula: $e = 0.0018d(n-1)$, it is supposed that its iron can bear, per square millimetre, an effort of $2^{kg}.870556$, since S is the effort per square millimetre, and $S = \dfrac{10334}{0.0036} = 2^{kg}.870556$.

The iron of a boiler in this supposition, supports then, per square millimetre, $2^{kg}.871$, while the elastic resistance of boiler iron is about 15 kg. per mm. sq. The coefficient of security then relatively to the limit of elasticity is $\frac{1}{5}$, and this sheet iron supports therefore, in fact, an effort morethan five times less than that which alters elasticity, and there would be nothing to fear in testing such aboiler even at a triple internal pressure, for then it would support, per square millimetre, an effort of only $8^{kg}.612$; comparing which with 15 kg., the limit of elasticity, we observe that only the 0.57 of this limit is attained. This value, 0.57, s not actually

reached, since the handbook formula makes not only the thickness of the boiler equal to $0^m0018\, d(n—1$t but increases this also by 0^m003.

A boiler is formed of leaves of sheet iron united by means of rivets. Let us consider a longitudinal



lap joint, assuming a diameter of rivet $= d$ and the interval between the rivets $= l$. What relation must exist between d and l in order that the yielding to rupture of the rivets and of the sheets of iron be the same? Rupture may take place either because the boiler iron be torn between two rivets or because the rivets be sheared. We demand, then, equal tendency to the sheets tearing and to the rivets shearing. But for sheet iron, the resistance to rupture by extension is assumed the same as the resistance of wrought iron to shearing. Therefore the section of the sheet iron torn must be equal to the section of the rivets sheared. But d being the diameter of a rivet, its section is $\pi\frac{d^2}{4}$, and the thickness of the sheet iron being e, the length that separates two rivets being l, the area of sheet iron to be torn is el. We must then have:

$$el = \pi\frac{d^2}{4}$$

This relation gives the suitable assemblage.

It may then be asked what space to allow between he rivets. Theoretically, there is an advantage in

making the rivets large, for thus is the continuity of the iron diminished less, but then, on the other hand, the joint may not be complete, and close l is generally made equal to $2d$, whence:

$$e2d = \pi \frac{d^2}{4}$$

$$8e = \pi d \quad d = e\,\frac{8}{\pi}$$

This boiler iron being rivetted, no longer offers a resistance proportional to l, but to:

$$\frac{l}{l+d} \text{ or, } \frac{k}{1+k} \text{ calling } l = kd$$

for we have:

resistance unrivetted, $\underset{l+d}{R}$: resistance rivetted, $\underset{l}{r}$ ::

or,

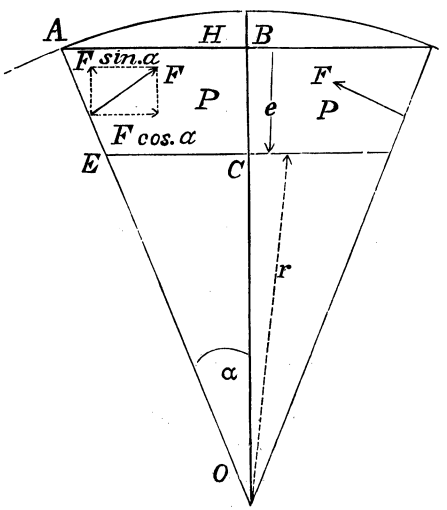
$$r\ (l+d) = Rl$$

$$r = R\ \frac{l}{l+d}$$

If we make $l = 2d$, $k = 2$ and $\frac{k}{1+k}$ becomes $\frac{2}{3}$, from which we see that resistance is diminished by $\frac{1}{3}$. Rivetting then weakens boiler iron about $\frac{1}{3}$, but this result is a superior limit, for the tightening of the sheets produces a friction that increases the tenacity generally, and the boiler's resistance then is not diminished by quite $\frac{1}{3}$. If, in testing a boiler, we reach 0.57 of the limit of elasticity, then, indeed, on the line of rivetting, $\frac{3}{2}$ 0.57 or the 0.86 of the limit is attained; but reasoning as we have just done, neglecting the term 0^m033 of the value e, it appears that in adding that margin we shall be in no danger of reaching the $\frac{3}{4}$ of the load that would injure elasticity.

DIMENSIONS TO BE GIVEN TO THE PIECES OF A BARRELLING, OR TUBBING OF A SHAFT.

Let us consider one of the sides of the wooden polygon forming any frame. A piece P is subjected to different forces enumerated as follows: it is pressed behind by a certain quantity of water, the



head of which is H. There are also acting two forces F decomposing into the forces $F\cos.\alpha$, and $F\sin.\alpha$, and, finally, the resistance of the piece opposed to these forces.

Let us consider the half of the piece ABC and suppose all in equilibrium. It is evident that we have some thing analogous to a prism fastened at one end (BC) and subjected at the other (AE) to an oblique force F.

Apply the theorem of moments. $F\cos.\alpha$ gives rise to a compression uniformly distributed $EA\frac{Z}{R}$

A being the section of the piece, Z being the ordinate of the neutral fibre to the centre of gravity, we have moreover: $A = eh$, e being the thickness of the piece and h the depth.

$$F\cos.\alpha = E\frac{z}{R}he$$

To calculate the moment of elasticity, we must know $F\cos.\alpha$ and $F\sin.\alpha$. But $F\sin.\alpha$ is equal to the thrust of the water, since all the forces are in equilibrium.

Therefore $F\sin.\alpha = HM1000$, M being the surface on which the pressure H is exerted, and 1000 being the weight of the unit of volume. But $M = hAB$, and as we have:

$$AB = BO\ tg.\alpha = (e+r)\ tg.\alpha$$

Finally,

$$F\sin.\alpha = 1000(r+e)tg.\alpha hH$$

Also, from this same equation, we find:

$$F\cos.\alpha = 1000(r+e)hH$$

But,

$$F\cos.\alpha = \frac{E}{R}hez$$

Therefore,

$$\frac{E}{R}hez = 1000(r+e)hH$$

$$\frac{E}{R}z = 1000\frac{r+e}{e}H$$

With this formula z may be determined if we know $\frac{E}{R}$

Now the moment of elasticity in the fixed section should be equal to the moment of the forces calculated from this section to the end, *i. e.*, we have;

$$\frac{E\mu}{R} = F\sin.\alpha\left(r+\frac{e}{2}\right)tg.\alpha$$

The second member is the moment of $F\sin.\alpha$. The moment of $F\cos.\alpha$ may be neglected, because the pieces bend very little; but we must insert the moment of the pressure of water, which is, since this force is uniformly distributed over AB:

$$1000(r+e)tg.\alpha hH\,\frac{r+e}{2}tg.\alpha$$

Replacing $F\sin.\alpha$ by its value:

$$F\sin.\alpha = 1000(r+e)tg.\alpha hH$$

we obtain:

$$\frac{E\mu}{s} = 1000(r+e)tg.\alpha hH\left(r+\frac{e}{2}\right)tg.\alpha$$

$$-1000(r+e)tg.\alpha hH\frac{r+e}{2}tg.\alpha$$

$$\frac{E\mu}{R} = 1000(r+e)tg^2\alpha h\ H\left((r+\frac{e}{2}\right)tg.\alpha - \frac{r+e}{2}$$

$$\frac{E\mu}{R} = 1000(r+e)tg^2\alpha hH\frac{(2r+e-r-e)}{2}$$

$$\frac{E\mu}{R} = 1000(r+e)tg^2\alpha hH\frac{r}{2}$$

In this expression μ is the moment of inertia of the section relative to an axis passing through the centre of gravity. We shall have then for the value of μ:

$$\mu = (\tfrac{4}{3}\,a^3b)$$

and applying in this case:

$$\mu = \tfrac{2}{3}\left(\frac{e}{2}\right)^3h$$

$$\mu = \tfrac{1}{12}e^3h$$

$$\frac{E}{R} = 12000(r+e)tg^2\alpha\,\frac{Hr}{2e^3}$$

$$\frac{E}{R} = 6000(r+e)tg^2\alpha H\frac{r}{e^3}$$

$\frac{E}{R}\left(z+\frac{e}{2}\right)$ is the fatigue at the point c, since it is a maximum strain. If in the expression which we have just obtained we replace $\frac{E}{R}$ by its value previously found, we obtain:

$$1000(r+e)\frac{H}{e} = 6000(r+e)tg^2a \; H\frac{r}{e^3}\, z$$

whence:

$$Z = \frac{e^2}{3rtg^2a}$$

consequently:

$$z+\frac{e}{2} = \frac{e^2}{3rtg2a}+\frac{e}{2} = \frac{2e^2+3retg^2a}{6rtg^2a}$$

We have found then $\frac{E}{S}$ and $z+\frac{e}{2}$. Multiplying the two values together, we shall have the maximum strain, which must not be greater than EI.

Therefore,

$$EI = \frac{6000(r+e)tg^2aHr}{e^3} \; \frac{2e^2+3retg^2a}{6rtga}$$

$$EI = \frac{1000(r+e)H(2e+rtg^2a)}{e^2}$$

$$e^2EI = 1000(r+e)H(2e+3rtg^2a)$$

$$e^2\left((EI)-2000H\right)-1000(2+3tg^2a)Hre$$

$$-1000H\,3r^2tg^2a = 0$$

This is an equation of the second degree with regard to e, which being resolved, gives the thickness of the piece. It has two roots, one positive, the other negative; the positive root only will be suitable.

From this equation we deduce:

$$e = \tfrac{1}{2}\frac{1000(2+3tg^2a)Hr}{EI-2000H}+\frac{1}{2(EI-2000H)}\times$$

$$\sqrt{\left(1000(2+3tg^2\alpha)Hr\right)^2+4000H\ 3r^2tg^2\alpha(EI-2000H)^2}$$

The thickness increases with the quantity H, and when considerable depths are attained, it varies rapidly, becoming infinite for $2000H = EI$.

If we make :

$$EI = 1{,}200{,}000$$

we find :

$$H = 600, \text{ for } e = \infty$$

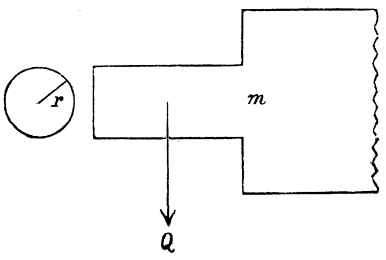
We notice that e is proportional to r, and also that as we multiply the number of sides $tg\alpha$ diminishes, and consequently e; therefore it will be advantageous to shorten the sides while increasing their number. The limit would be a circle, and when the tubbing is formed of castings it is given a circular section.

JOURNAL OR TRUNNION.

Let us consider now, an axle on which is fixed a gearing or a water wheel, and first study the trunnion or journal. This is submitted to a certain load from the weight of the axle and the effort exerted on this axle. This weight and effort may be uniformly distributed or not on the journals, and will be for example a number n of horse powers.

Let, for instance, R be the radius of a cog-wheel on this axle, $R\omega$ is its velocity at the circumference, if Q' be the force, $R\omega Q'$ is the work transmitted, and we have :

$$R\omega Q' = n\ 75 \text{ kgm.}$$



consequently:

$$Q' = \frac{n\ 75}{R\omega}$$

And let the weight of axle $= Q''$

We are supposed to know at what point the force Q' is applied, and therefore by the decomposition of these forces, we may determine the load upon each journal. Let us call any such load Q. The journal rests on a surface of contact over which Q is uniformly distributed, so that its point of application is the middle of the length. If l be the length, we shall have:

$$pl = Q$$

p being the load on the unit of length, and this evidently is the case of a solid fastened at its extremity, m, and supporting a load uniformly distributed over it; wherefore:

$$\frac{E\mu}{R} = p\,l\,\frac{l}{2} = Q\,\frac{l}{2}$$

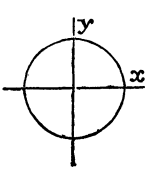
$$\frac{E}{R} = \frac{Ql}{2\mu}$$

The maximum strain exists in the fastened section, and we have for its value:

$$\frac{EV}{R} = \frac{Ql}{2}\,\frac{V}{\mu}$$

Now as the journal is a cylinder, μ is the moment of inertia of a circle relative to its diameter equal to

$$\int \omega y^2$$



In order to oblain a value for this last expression, we may remark that:

$$\int \omega y^2 + \int \omega x^2 = \int \omega(x^2 + y^2)$$

but,

$$\int \omega y^2 = \int \omega x^2$$

since, in a circle everything is symmetrical relative to any diameter; hence:

$$\int \omega y^2 = \tfrac{1}{2} \int \omega(x^2 + y^2$$

i. e., the moment of inertia of a circle relative to any diameter is half its moment of inertia relative to an axis passing through its centre and perpendicular to its plane.

Now, $x^2 + y^2 = r^2$, therefore ·

$$\int \omega y^2 = \tfrac{1}{2} \int \omega r^2$$

The element ω at the distance r, is:

$$2\pi r \, dr$$

and its moment of inertia is:

$$2\pi r dr \, r^2$$

The moment entire $\int \omega r^2$, is then:

$$2\pi \int_0^r r^3 dr = 2\pi \frac{r^4}{4}$$

and this being equal to $2 \int \omega y^2$, we have:

$$\int \omega y^2 = \tfrac{1}{4} \pi r^4$$

Now:

$$V = r$$

wherefore,

$$\frac{V}{\mu} = \frac{4}{\mu r^3}$$

and:

$$\frac{EV}{R} = \frac{Ql}{2} \frac{4}{\pi r^3}$$

And as the strain must not exceed Ei, we may write:

$$Ei = \frac{Ql}{2} \frac{4}{\pi r^3} = Ql \frac{2}{\pi r^3}$$

Ei is a quantity given for the material employed. The value of r is:

$$r^3 = \frac{2}{\pi} \frac{Q}{Ei} l$$

or, if D be the diameter of the trunnion or journal,

$$D^3 = \frac{16}{\pi} \frac{Q}{Ei} l$$

The diameter must then increase with the cube root of l for a given load of Q.

Generally l is made $= KD$, K being comprised between 1 and 2. Substituting this in the value of D, we obtain:

$$D^2 = \frac{16}{\pi} \frac{Q}{Ei} K$$

According to this formula, if K be constant, the diameter is proportional to the square root of the load. When the axle turns the journal strains considerably, and moreover after it has turned a half revolution, a point of the strained section which was first subjected to an effort of compression is now subjected to one of extension, and these changes of action make the stress more injurious. For a journal of forged iron we may assume:

$$Ei = 2{,}000{,}000$$

the load per square millimetre will not then exceed 2 kilo., and as this is $\frac{1}{7}$ of the elastic resistance of iron, the coefficient of security is $\frac{1}{7}$. We should employ a somewhat smaller number for castings, and it would be expedient to increase the diameter a little; however often enough the same formula is taken:

$$D^2 = \frac{16}{\pi} \frac{QK}{2{,}000{,}000} \frac{Q}{\frac{\pi}{16} 2{,}000{,}000} K$$

$$D^2 = \frac{1}{\frac{(3.141593)}{8}} KQ \quad D^2 = \frac{1}{392700} KQ$$

In varying circumstances, Ei may be modified and the formula may vary from:

$$D^3 = \frac{1}{350000} Ql$$

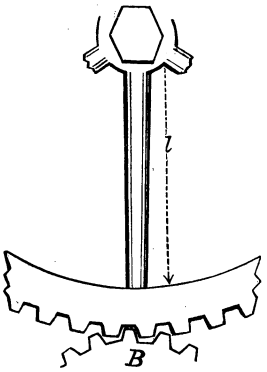
to,

$$D^3 = \frac{1}{700000} Ql$$

corresponding to the values:

$$Ei = 1^{kg}.783 — Ei = 3^{kg}.567$$

Now we will examine the case of the cog-wheel on the axle, beginning with the effort which is exerted on a radius of the wheel. Let us have, for example, a hexagonal axle, on which a cast iron collar supports the arms. We are supposed to know the force to be transmitted by the gearing, but on account of the imperfect solidarity of the system we are not certain how this force is distributed; each arm would support $\frac{1}{6}$ of the pressure to be transmitted if the solidarity were perfect. But suppose the most unfavorable circumstances, and that the arm nearest the gearing B supports the whole effort. We have then a solid fastened at one end and supporting a given



force at the other; therefore the formula applicable is:

$$\frac{E\mu}{R} = Q(l-x)$$

l being the length of the arm from the circumference of the collar to the rim of the wheel:

$$\frac{Ev}{R} = Ei = Q(l-x)\frac{V}{\mu}$$

What first would be the form to give the arm for a solid of equal resistance? Its section must evidently diminish from the collar to the jant, and if the sections be similar, μ will be of the form KV^4; V being a dimension homologous in all sections, since the moment of inertia is of the 4th degree; $\frac{V}{\mu}$ is then of the form $\frac{1}{KV^3}$, and consequently:

$$Ei = Q(l-x)\frac{1}{KV^3}$$

whence:

$$V^3 = \frac{1}{K}\,\frac{Q(l-x)}{Ei}$$

The homologous dimensions then must decrease with the ordinates of a cubic parabola whose vertex is at the end of the arm. Generally the section of wrought and cast iron arms is rectangular, and the larger dimensions are of course set in the plane of

the gearing where the efforts are. Designating by $2a$ and $2b$ the dimensions of a rectangle we have:

$$\mu = \tfrac{4}{3}a^3b$$

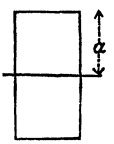
$$\frac{a}{\mu} = \frac{3}{4a^2b}$$

$$Ei = Q(l-x)\,\frac{3}{4a^2b}$$

$$a^2b = \tfrac{3}{4}Q\,\frac{(l-x)}{Ei}$$

If we consider the fastened section $x = 0$ and $a^2b = \frac{3}{4}\,\frac{Ql}{Ei}$

This gives the area of the fastened section, and if the arms are made constant in section, the strain will be less for all other points.



Between a and b, there may exist relations which experience dictates; for instance, in cast iron, $b = \frac{1}{5}a$, and consequently:

$$a^2b = \frac{a^3}{5}$$

whence:

$$\frac{a^3}{5} = \tfrac{3}{4}\,\frac{Ql}{Ei}$$

or,

$$a^3 = \tfrac{15}{4}\,\frac{Ql}{Ei}$$

Any arm is more or less supported by the solidarity existing between the different arms; and moreover, the efforts on it do not vary in direction: wherefore we may call $Ei = 7{,}500{,}000 = 7^{kg}.5$ per millimetre square, and find:

$$a^3 = \tfrac{15}{4}\,\frac{Ql}{7{,}500{,}000} = \frac{Ql}{2{,}000{,}000}$$

If the force be introduced in terms of h. power, we shall have:

$$Q(l+r)\omega = 75n$$

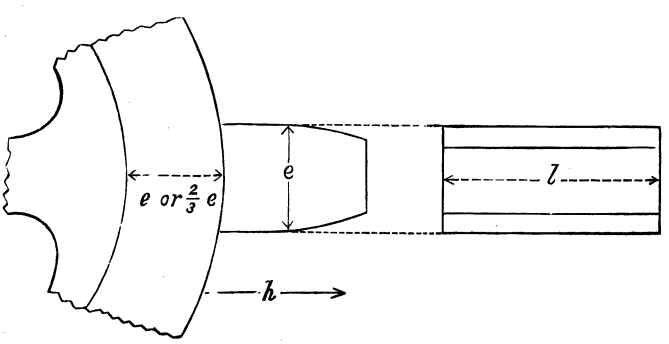
r being the radius of the collar and n being the number of hp.;

$$a^3 = \frac{75nl}{2{,}000{,}000(l+r)\omega}$$

When we consider a series of cog-wheels which are to transmit the same quantity of work, but accelerating in proportion as ω for each may augment, it is evident that a may be diminished in the accelerating wheels. a varies inversely as the cubic root of ω and directly as $\frac{l}{l+r}$. When the gearing, however, is large, $\frac{l}{l+r}$ is near 1; and then a may be considered as varying sensibly in inverse ratio with the cubic root of ω.

DIMENSIONS OF THE TEETH OF THE WHEEL.

Any tooth may be considered as a solid fastened at one end and loaded successively upon different points of its length. Let e be the thickness of the tooth on the pitch circle, h the radial length of the tooth, and l its width. There may be admitted a



certain proportion between these quantities: for instance, if $e = 1$, it is suitable that $h \overset{=}{<} \frac{3}{2}$ and $l \overset{=}{<} 5$.

The breadth of the jant at the utmost, is equal to e or $\frac{2}{3}e$ when it is reinforced by a flange at the joining with the arms.

When two teeth are just separating, we shall have:

$$\frac{E\mu}{R} = hQ$$

$$Qh\frac{V}{\mu} = Ei$$

$$\mu = \tfrac{4}{3}a3b$$

but,

$$a = \frac{e}{2} \text{ and } b = \frac{l}{2}$$

whence,

$$\mu = \tfrac{4}{3}\,\frac{e^3 l}{16} = \frac{e^3 l}{12}$$

moreover,

$$V = \frac{e}{2}$$

consequently,

$$Qh\frac{6}{e^2 l} = Ei$$

whence:

$$e^2 l = 6\,\frac{Qh}{Ei}$$

or, by making $h = \frac{3}{2}e$ and $l = 5e$:

$$e^2 = 1\ 8\frac{Q}{Ei}$$

Replacing Q by its value hp., we shall find:

$$e^2 = 1\ 8\frac{1}{Ei}\ \frac{\pi\ 75}{R\omega}$$

As e depends on $R\omega$, two teeth interlocking must have the same thickness; but if two wheels are fixed on the same axle, one receiving the force, and the other transmitting it to another axle, the larger wheel may have the smaller teeth.

Teeth of wheels are subjected to an effort always in the same direction; moreover, as there are more than two teeth in contact at once, this effort is distributed usually over several; but as wear may prevent teeth from touching simultaneously, we may consider that fact, and call:

$$Ei = 1{,}500{,}000 = 1^{kg}.5 \text{ per mm. sq.}$$

For wooden teeth:

$$Ei = 600{,}000 = 0^{kg}.6 \text{ per mm. c.}$$

For castings and wrought iron:

$$e^2 = \frac{Q}{800{,}000}$$

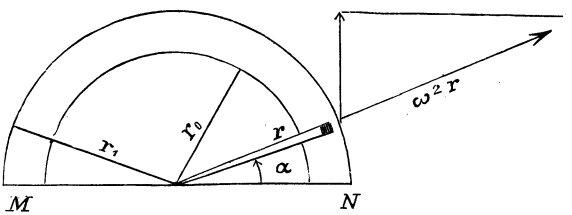
FLY WHEELS.

The jant of a fly wheel is a cast iron annulus attached to the axle by arms. Rupture of the jant might take place on account of the centrifugal force at the circumference; and let us now imagine it as occuring along a horizontal diameter. An element comprised between 2 radii, making the angles α and $\alpha + d\alpha$, and the radii r and $r + dr$, l being the breadth of the jant, has for expression of its volume $r\,dr\,d\alpha\,l$ and $\frac{\delta}{g} r\,d\alpha\,dr\,l$ is its mass; δ being the weight of a unit of volume. $\frac{\delta}{g} r\,d\alpha\,dr\,l\,\omega^2 r$ is then

the centrifugal force of the elementary mass. The component perpendicular to the plane MN where rupture occurs is:

$$\frac{\delta}{g} r\, da\, dr\, l\omega^2 r \sin.a$$

This is applied to the element $dr\, da$ and the total force above the plane MN will be:



$$\frac{\delta}{g} l\omega^2 \int_{r_0}^{r_1} \int_0^{\pi} r^2 dr \sin.a\, da = \tfrac{2}{3} \frac{\delta}{g} l\omega^2 (r_1^3 - r_0^3)$$

Such is then the expression of the force tending to break a fly wheel along a diameter. Rupture is resisted by the arms, and the tenacity of the jant itself. Neglecting, however, the resistance of the arms and calling l the breadth of the jant, we must have:

$$\tfrac{2}{3} \frac{\varrho}{g} l\omega^2 (r_1^3 - r_0^3) = Ei\, 2l(r_1 - r_0)$$

which being reduced, becomes:

$$Ei = \tfrac{1}{3} \frac{\varrho}{g} \omega^2 (r_1^2 + r_1 r_0 + r_0^2)$$

Ei being assumed, the other are determinable.

Call r the mean radius of the jant and e the half thickness: then,

$$r_1 = r + e \qquad r_0 = r - e$$

$$Ei = \tfrac{1}{3} \frac{\delta}{g} \omega^2 (3r^2 + e^2)$$

$$Ei = \frac{\delta}{g}\omega^2 r^2\left(1+\frac{e^2}{3r^2}\right)$$

$$Ei = \frac{\delta}{g}v^2\left(1+\tfrac{1}{3}\frac{e^2}{r^2}\right)$$

in which v is the velocity at the mean circumference of jant. The strain depends then upon v^2 and upon $\frac{e^2}{r^2}$; increasing with the velocity and the thickness of the jant. If e be very small compared to r, we may write:

$$(a)\ Ei = \frac{\delta}{g}v^2$$

If we give e its maximum value, *i. e.*, $e = r$, we will have:

$$(b)\ Ei = \tfrac{4}{3}\ \frac{\delta}{g}\ V^1 2$$

V^1 being the velocity at the mean circumference of wheel. But $V^1 = \frac{V}{2}$: V being the velocity at the outer circumference, whence:

$$(c)\ Ei = \tfrac{1}{3}\frac{\delta}{g}\ V^2$$

In general, the velocity of a fly wheel ought not to exceed 30^m, 40^m, 45^m per second.

Let $v = 30^m$, then $v^2 = 900$, $g = 9.81$, $\delta = 7200^{kg}$, and we thus would have assumed:

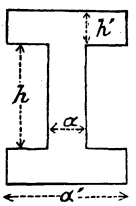
$$Ei = 660{,}000 = 0^{kg}66 \text{ per mm. c.}$$

In case castings act by extension, the load in practice ought not to exceed 2^{kg} to $2\frac{1}{2}^{kg}$, and as a rupture by extension is here in question, we see that a fly wheel with a velocity of 60^m per second, would be as heavily loaded as is at all prudent. At a velocity of 120^m, the wheel would undergo a stress

of $10^{kg}.56$ per square millimetre, and as the actual rupturing force for castings is from 11 to 13 kg., it would be in danger of immediate rending. If the jant be well centred its centrifugal force will not affect the axle or the journals, but if it be not perfectly centred, a pressure is generated rapidly variable in direction, and in the case of an uncentred crank, for instance, in a locomotive at a great velocity, these efforts may be reckoned by thousands of kilogrammes.

BALANCE BEAM.

The balance beam may be considered as fixed at the middle and subjected to two forces applied one to each end. As each end is alternately subjected to an effort in opposite directions, the section of the

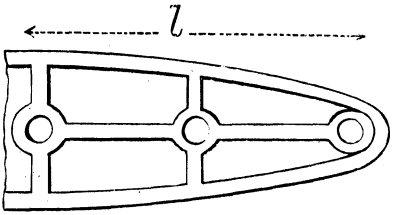


beam is generally a double T. Ordinarily we may comprise a between $0^{m}.02$ and $0^{m}.07$, h between $6a$ and $16a$. h' is given the same value as a, and a' is comprehended between $3a$ and $4a$.

To determine the dimensions then of the balance beam's cross section, we make the hypothesis $EI = 2{,}000{,}000$, and calculate simply the area of the rectangle ah. To this in the construction, we add two flanges with the dimensions indicated. In this manner Ei is very much less than 2,000,000. The section being calculated for the fixed point, the beam may be diminished towards the end as has

already been described to obtain the solid of equal resistance.

The half length l of the beam is determined by the geometric conditions of transmission. It is com-



prised between 1·5 and 2 the stroke of the piston.

To calculate ah we have as always :

$$\frac{EV}{R} = \frac{QlV}{\mu}$$

but,

$$\mu = \tfrac{1}{12}\, h^3 a \quad \text{and} \quad V = \frac{h}{2}$$

whence,

$$\frac{EV}{R} = \frac{Ql\,\frac{h}{2}}{\frac{1}{12}\, h^3 a} = \frac{6Ql}{h^2 a} = EI \quad \text{or} \quad h = \sqrt{\frac{6Ql}{aEI}}$$

The ratio between h and a is very variable in practice ; call it k, then we will have :

$$\frac{6Ql}{kh^3} = EI$$

or,

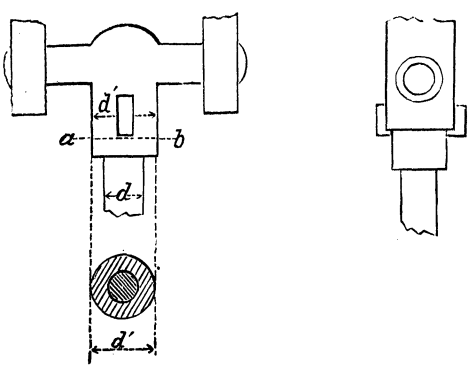
$$h = \sqrt[3]{\frac{6Ql}{kEI}}$$

As each trunnion must support a load equal to $\frac{1}{2}$ the weight of beam, plus the pressure at the end of the beam, let us call those efforts Q and calculate the diameter as has already been indicated, neglecting the torsion, which in such cases is small enough to be left out of account.

PISTON ROD.

In low pressure engines the effort on the piston being 10334^{kg} per m.^{2} we may assume according to Watts' practice the tenth of the diameter of the piston for the diameter of the rod. That corresponds to $1^{kg}.033$ per square millimetre on the piston rod, and its diameter should be increased to one-fifth in machines at 4 atmospheres. The rod is fixed to the cross-head of the parallelogram, the length of whose shorter side is equal to that of the crank.

In order that the effort be well distributed and the strain be the same on the piston rod and the socket of the cross head, it is evident that in a section as



ab the area of the socket must be equal to that of the piston rod. The section of the rod is:

$$\pi \frac{d^2}{4}$$

The section of the socket is:

$$\pi \frac{d_1{}^2 - d^2}{4}$$

therefore:

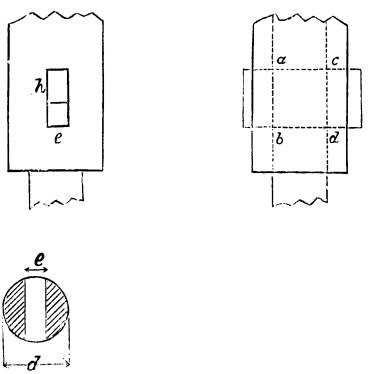
$$\pi \frac{d^2}{4} = \pi \frac{d_1{}^2}{4} - \pi \frac{d^2}{4}$$

whence:

$$d_1{}^2 = 2d^2$$

$$d^1 = d\sqrt{2} = \text{diameter of socket.}$$

The joint might also be destroyed by the rupture of the key by shearing. The key resists by its two faces ab and cd; the resisting section of the key is then $2eh$, e being the thickness of the key, and h its depth. If we consider the piston rod at this



same point, as its coefficient of resistance is the same as that of the key, the two sections would have to be the same, but the section of the rod is $\pi \frac{d^2}{4} - ed$, we need:

$$2eh = \pi \frac{d^2}{4} - ed$$

If this equality subsists, the key strains as much as the rod. In general, $h = 4e$ or $6e$.

Let us call $h = ke$; we shall then have:

$$2ke^2 = \pi \frac{d^2}{4} - ed$$

and giving any value to k we may therefrom deduce e.

$$2ke^2 + de - \pi \frac{d^2}{4} = 0$$

$$e = -\frac{d}{4k} \pm \sqrt{\frac{d^2}{16k^2} + \frac{\pi d^2}{8k}}$$

Neglecting the minus sign before the radical, which would give e negative, we have:

$$e = \sqrt{\frac{d^2}{16k^2} + \frac{\pi d^2}{8k}} - \frac{d}{4k}$$

$$e = \frac{d}{4k}\left(\sqrt{1 + \frac{2\pi}{k}} - 1\right)$$

This supposes that the key is of the same material as the rod. If the key be of steel and the rod of iron, we should write $2eh = m\left(\pi\frac{d^2}{4} - ed\right)$; m being the relation of the resistance of steel to that of iron. If we make $k = 6$, we find:

$$e = 0,\ 2\ d$$

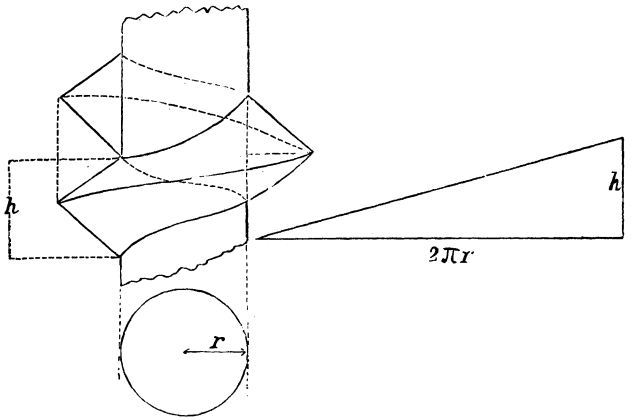
The weakening of the rod that results from keying it is given by the ratio:

$$\frac{\frac{\pi d^2}{4} - ed}{\pi\frac{d^2}{4}} = 1 - \frac{4ed}{\pi d^2} = 1 - \frac{4e}{\pi d}$$

If $e = 0,\ 2\ d$, we find:

$$\text{Ratio} = 1 - \frac{4(0\ 2)}{\pi} = 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

In order to obviate this inconvenience, the section of the rod is increased at the key, and the key itself is placed on edge in such way as to diminish e while rapidly increasing h. Again, the piston rod may be connected with the cross head by a screw or double screw. Rupture may then occur by the stripping of the threads or else by the rupture of the screw. If



h be the rise of one of the helices for every turn of the screw, the surface of stripping for one turn will be:

$$2\pi r\ h$$

and designating by n the number of threads, we shall have for the entire resisting surface:

$$n\ 2\pi r\ h$$

and admitting that the resistance to rupture by shearing is the same as by extension, we may write:

$$\pi r^2 = 2\pi r\ h\ n$$

Thence we may derive:

$$r = 2hn$$

If we consider the two links of the parallelogram that unite the rod with the balance beam, we are led to give them each a section half that of the rod; but in order to obviate the irregularities of distribution that might take place, they may each have a section equal to $\frac{3}{4}$ of the section of the rod. The other sides of the parallelogram have no strain on them worth considering.

CONNECTING ROD.

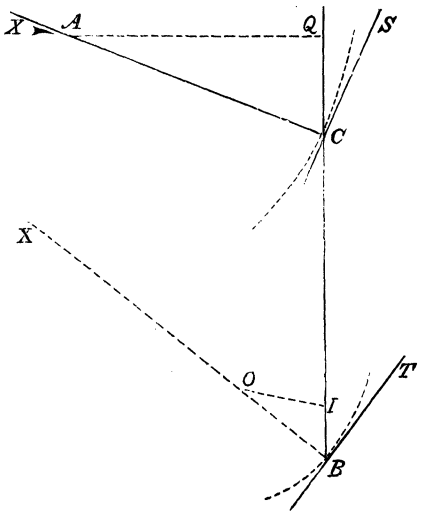
If we consider a connecting rod of cast iron it may be described as follows: Firstly it is reinforced with a bulge, at the middle, as is suitable for all bodies of a certain length subjected to efforts of compression. In order to obviate the strains resulting from the lashing of the piece, the effort in the middle must not be greater than $0^{kg}.3$ or $0^{kg}.4$ per square millimetre. The bulge increases the solidity, and may, on occasion, be strengthened by a salient flange. The section may be circular or elliptical, which last still increases the rigidity of the piece.

When the machine is direct acting and the connecting rod short and of forged iron, the swelling is made near the attachment to the fly wheel, for there will be found the maximum fatigue, since the other end of the rod moves usually on slides, and inertia is most in play near the crank. A connecting rod of forged iron may bear a load greater than $0^{kg}.3$ or $0^{kg}.4$ per millimetre square, but it should not be submitted to the same load as the piston rod, because it strains more, and is subjected alternately to efforts of compression and of extension.

The transverse dimensions of the connecting rod are determined by the variable effort which it is to transmit. In low pressure engines, according to Watt, the rod may be of cast iron and its middle section be $\frac{1}{28}$ that of the piston, its extreme sections $\frac{1}{35}$ of the same, which corresponds to loads of $0^{kg}.35$, and $0^{kg}.28$ per millimetre^2. In high pressure engines, however, forged iron is preferred, and the load may vary from $0^{kg}.5$ to $0^{kg}.6$ at the middle, and from $0^{kg}.9$ to $1^{kg}.10$ at the ends.

Let us determine the tension of a connecting rod which transmits to a crank the oscillations of a balance beam. Let A be the projection of the axis of the beam and O the same of the crank. Let CB

be the position of the connecting rod at a given moment. Call l the length of the rod, b the half



length AC of the beam, ϱ the radius of crank, OB, ω the angular velocity of this last, and ω' the angular velocity of beam.

Now there is a simple relation between these two velocities, for the velocities of the points B and C being in the direction of the tangents BT and CS, the centre of instantaneous rotation of the system is at the intersection of the lines BX and CX, and the two velocities in question are as the lines BX and CX. But drawing OI parallel to AC, the triangle OIB will be similar to XCB and the two velocities are consequently proportional to the sides OB and OI, whence we infer:

$$\omega\varrho : \omega' b :: OB : OI :: \varrho : \lambda$$

calling λ the length OI, whence:

$$\omega' = \omega \frac{\lambda}{b} \quad (1)$$

This being established, let F be the motor force at the other end of the beam, a its distance from A,

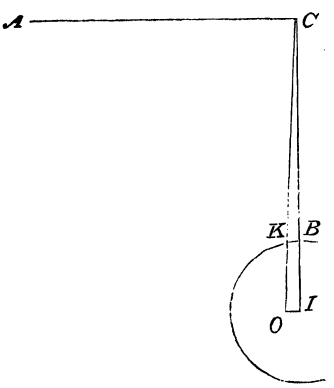
let T be the tension or pressure exerted on the beam by the connecting rod, q the distance AQ of this force from the axis, M the mass of the beam, and R its radius of gyration, we have then the angular acceleration :

$$\frac{d\omega'}{dt} = \frac{Fa - Tq}{MR^2} \quad (2)$$

but from $eq(1)$ we have, by differentiating and considering ω as a constant (which we may do, since the motion of the crank is sensibly uniform) :

$$\frac{d\omega'}{dt} = \frac{\omega}{b}\frac{d\lambda}{dt} \quad (3)$$

The greatest variation of λ is in the positions cor responding to the dead points. Let CK be the direction of the connecting rod at such a moment and CB a position infinitely near. The infinitely



small right line OI parallel to AC is the value of $d\lambda$, since λ is 0 at the dead point. Call dx the small arc KB, we have :

$$dx = \omega\varrho dt$$

or,

$$dt = \frac{dx}{\omega\varrho}$$

Substituting this value in *eq.* (3), we have:

$$\frac{d\omega'}{dt} = \frac{\omega^2 \varrho}{b} \frac{d\lambda}{dx} \quad (4)$$

But in the triangle COI, we find:

$$OI : CC :: \sin.C : \sin.I$$

whence:

$$d\lambda = (l+r) \frac{\sin.C}{\sin.I}$$

and in the triangle CKB, which may be considered rectilinear and rectangular at K, we have:

$$KB = CK \text{ tang.} C$$

or,

$$dx = l \text{ tang.} C$$

consequently,

$$\frac{d\omega'}{dt} = \omega^2 \frac{\varrho}{b} \frac{l+\varrho}{l} \frac{\sin.C}{\text{tang.}C} \frac{1}{\sin.I}$$

The $\sin.C$ may be assumed $=$ tang.C and the angle L may be considered to have the same sin. as the angle ACI or ACO, wherefore:

$$\frac{d\omega'}{dt} = \omega^2 \frac{\varrho}{b}\left(1+\frac{\varrho}{l}\right)\frac{1}{\sin.ACO} \quad (5)$$

Taking the value of Tq from *eq.* (2), and replacing this value of $\frac{d\omega'}{dt}$ we find:

$$Tq = Fa - MR^2\omega^2 \frac{\varrho}{b}\left(1+\frac{\varrho}{l}\right)\frac{1}{\sin.ACO}$$

Observing finally that $b\sin.ACO$ or $AC\sin.ACO$ is precisely equal to the distance q of the point A from the direction OC, we deduce:

$$T = F\frac{a}{q} - MR^2 \frac{\omega^2 \varrho}{q^2}\left(1+\frac{\varrho}{l}\right) \quad (6)$$

and, from this value, the section of the connecting rod may be calculated. If, for instance, it be required that the rod be loaded with not more than 1,000,000 ^{kg} per m.^{2}, we may write:

$$1{,}000{,}000\ \tfrac{1}{4}\pi d^2 = T$$

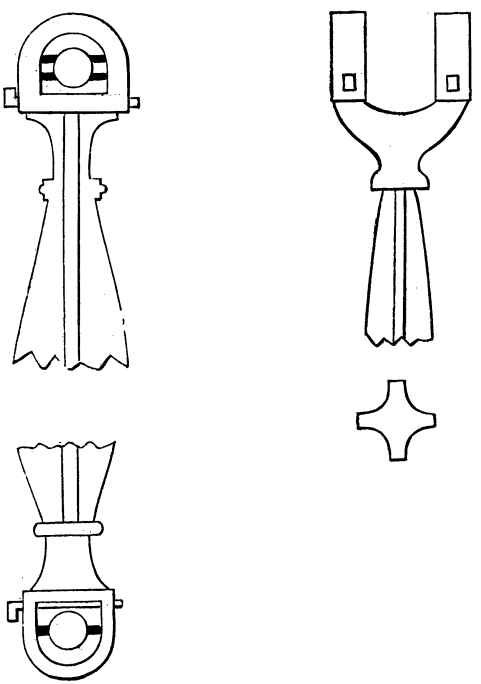
or,

$$d = \tfrac{1}{500}\sqrt{\frac{T}{\pi}}$$

In case of direct connection, where the connecting rod articulates directly with the piston rod, we may, in a similar manner, find

$$T = F - M\omega^2\varrho\left(1+\frac{\varrho}{l}\right)\frac{1}{\sin.I}$$

The connecting rod is generally five times as long as the crank. Besides the ordinary forms of cross section, they are sometimes made hollow cylinders, a form recommended by Hodgkinson. The upper



end connects with the balance beam by a fork through which and the beam passes an axis resting on bearings which can be tightened by keys, one moveable and the other fixed.

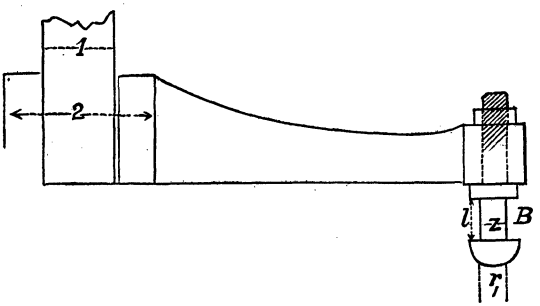
CRANK.

The button B of the crank is in the same condition as a journal or trunnion. Let the tension of the connecting rod Q, be the effort on it, we have:

$$\frac{Ql}{2} = \frac{E\mu}{R}$$

and,

$$Ei = \frac{Ql}{2} \frac{V}{\mu}$$



But, $V = r_1$; and μ is the moment of inertia of a circle relative to one of its diameters, wherefore:

$$\mu = \frac{\pi r_1^4}{4}$$

and,

$$\frac{V}{\mu} = \frac{r_1 4}{\pi r_1^4} = \frac{4}{\pi r_1^3}$$

$$\underline{Ei} = \frac{Ql}{2} \frac{4}{\pi r_1^3} = \frac{2}{\pi} Q \frac{l}{r_1^3}$$

If we assume $Ei = 300{,}000$, $\frac{l}{r^3}$ may be determined.

If now we consider the crank itself, it is evidently subjected sometimes to an effort of traction, sometimes to an effort of extension, and sometimes to an effort applied to its extremity perpendicularly to its length, or, in other words, to an effort of flexion. Let r be the radius of the crank. When the connecting rod is at right angles with the crank, this last is in the condition of a solid body, fastened at one end and loaded at the other, wherefore:

$$Qr = \frac{E\mu}{R}$$

and,

$$Ei = Qr\frac{V}{\mu}$$

We may assume that the crank be of rectangular section, and that it be determined in the same manner as the solid of equal resistance loaded at one end; we have generally, then:

$$V = \frac{a}{2}$$

and,

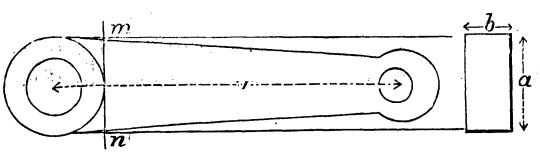
$$\mu = \tfrac{1}{12}\, a^3 b$$

whence,

$$\frac{V}{\mu} = \frac{6}{a^2 b}$$

and,

$$Ei = 6Q\frac{r}{a^2 b}$$



If the crank button and crank be of the same material, the two values of Ei just found, should be equal, that is to say:

$$\frac{2}{\pi} Q \frac{l}{r_1^3} = 6Q\frac{r}{a^2 b}$$

whence:

$$\frac{2}{\pi}\,\frac{l}{r_1^3} = \frac{6r}{a^2 b}$$

or,

$$\frac{l}{r_1^3} = 3\pi\,\frac{r}{a^2 C}$$

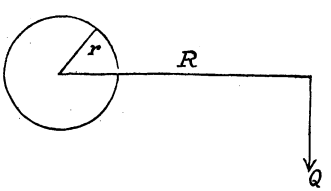
AXLE OF A FLY WHEEL, WATER WHEEL OR TURBINE.

This axle is subjected to a determinate force ap-applied to the end of a radius R. We know that at the point most strained by torsion in any section whatever, we have:

$$Ei = QR\frac{r}{\mu}$$

wherefore as $\mu = \frac{\pi r^4}{2}$

$$Ei = QR\,\frac{2}{\pi r^3}$$



Let the engine be of N horse power, making n revolutions per second; at every piston stroke a work $Q4R$ is produced, and the work for n turns is:

$$Qn4R$$

whence:

$$Qn4R = 75\,N$$

$$QR = \frac{75}{4}\,\frac{N}{n}$$

and,

$$Ei = \frac{75}{2\pi} \frac{1}{r^3} \frac{N}{n}$$

or,

$$r^3 = \frac{75}{2\pi} \frac{1}{Ei} \frac{N}{n}$$

whence:

$$d^3 = \frac{300}{\pi} \frac{1}{Ei} \frac{N}{n}$$

If d be expressed in centimetres and n be the number of revolutions per minute, we will have:

$$d^3 = \frac{1{,}000{,}000 \quad 300 \quad 60}{\pi} \frac{1}{Ei} \frac{N}{n}$$

If $Ei = 1{,}000{,}000$,

$$d^3 = 6000 \frac{N}{n}$$

or,

$$d = 10 \sqrt[3]{\frac{6N}{n}}$$

Instead of the coefficient 6000 in the value of d^3, variable coefficients may be employed from 4,300 up to 6,900, which correspond to different values of Ei. In proportion as the machine be more powerful, d will increase, and in proportion as it is more rapid, d will diminish.

This formula relates to axles of cast iron. For an axle of forged iron d, we may write:

$$d_1^3 = \tfrac{2}{3}\, d^3$$

or,

$$d^1 = 10 \sqrt[3]{4 \frac{N}{n}}$$

As an example, if $N = 50$, $n = 25$, we shall find:

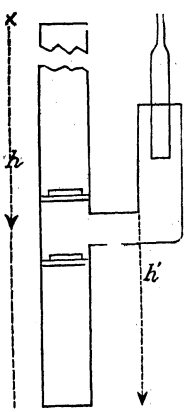
$$d^1 = 20^{cm}.$$

and d^1 will still be the same number as long as the ratio $\frac{N}{n}$ does not change; if, for instance, $N = 100$ and $n = 50$.

If the machine has a cut-off, a somewhat different result would be found, but most frequently the same formula is applied.

PUMP ROD.

If A be the section of the rod of a force pump, it will support an effort $AEI = F$. Let P_0 be the atmospheric pressure, h the height of the water level above the piston, and h' the height of the piston above lower level, and P the weight of cubic metre



of water $= 1000^{kg}$. The piston or plunger experiences from the lifted column a pressure $P_0 + Ph$ per unit of surface, and from below is experiences the pressure $P_0 - Ph'$. The effective pressure, then, which it supports, is $P(h+h')$, or PH calling H the difference of levels, and if ω be the area of piston, the entire pressure supported by the rod is:

$$\omega PH = F = AEI$$

As example, suppose $H = 21^m$ and $A = 0^m.04$, then:

$$\omega PH = 840^{kg}$$

and if the rod be of wrought iron, we may assume

$$EI = 2{,}500{,}000$$

whence,

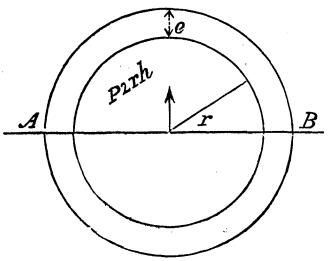
$$A = 0{,}000{,}336$$

and if x represents the diameter,

$$x = 0{,}020{,}068$$

A main-rod of a system of mining pumps would have to support at any point the pressure on all the plungers below that point. The section at the top would support a column equal to all the lifts; but as the weight of the main rod, thus calculated, would in construction, generally more than balance the weight of the column, it is found simply in the condition of a prism supporting its own weight, and may be calculated accordingly. Under this consideration the rod is subjected to extension, and EI per millimetre^2 = $0^{kg}.6$; if the rod be considered when it sinks, it is subjected to compression and EI per millimetre^2 = $0^{kg}.4$. If the pump be a lift pump, the rod supports at a given time both its own weight and all that of the water lifted by it.

If the rod be horizontal or inclined, a certain effort of flexure is introduced which can be considered as in ordinary cases, the flexure being due to the weight of rod acting at a point half way between any two supports. A lift-pipe tends to rupture in a plane



through AB. The pressure at any section is $P2rh$, and on the surface, proportional to dh, is $P2rhdh$;

the resistance is $EI\ 2edh$; equalizing and integrating, we have:

$$EI\ 2eh = rPh^2$$

or,

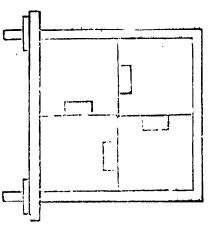
$$e = \frac{1}{2}\ \frac{Phr}{EI}$$

in which,

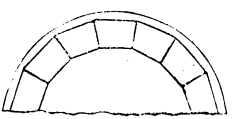
$$EI = 200{,}000$$

CROSS STRAIN OF AXLE TREES FOR WATER WHEELS.

Wooden axle trees are generally square in section, though sometimes polygonal or circular, usually also they are made of a single piece, but occasionally of



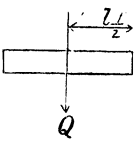
several united by iron collars or frets. The axles of the large Cornish water wheels are shaped into prisms of 12 to 16 sides, and they are also strengthened by means of segments juxtaposed on the axle to increase its diameter, particularly where the wheel bears. These segments are clamped by iron hoops put on hot and bring the axles sometimes to five feet in diameter. The journals are forged or cast iron and in an important axle are attached in the



following manner: The axle is terminated by a slightly conical trunk and a muff or shoe carrying the journal, is fixed to the truncated part by a central spike which penetrates the axle, and by wings also, slotted into the exterior and reinforced by hoops or tires, put on hot, to shrink. The wings are tightened by wedges, and iron wedges are also driven if need be into the wood at the axle to increase its solidity. Sometimes the journal is simply carried by a cylindrical shoe wedged firmly on the end. If the axle be of cast iron and solid the journal is cast with it; if it be hollow, it is terminated by a bridle on which the journal may be bolted.

Solid axles are round or polygonal; if they are hollow, they are made cylindrical.

Suppose a cylindrical axle carrying a wheel at its



middle point, calling Q the weight of wheel and l the length of axle, we shall have:

$$EI = \tfrac{1}{4}\frac{VQl}{\mu}$$

but,

$$\mu = \tfrac{1}{4}\pi r^4$$

and,

$$V = r$$

wherefore,

$$EI = \frac{Ql}{\pi r^3}$$

and,

$$r = \sqrt[3]{\frac{Ql}{\pi EI}}$$

If the axle be hollow, r being the exterior and r'

the interior radius, we shall still have:

$$V = r$$

but,

$$\mu = \tfrac{1}{4}\pi(r^4 - r_1^4)$$

It is convenient to make $r' = \frac{3}{5}r$, whence:

$$\mu = \tfrac{1}{4}\pi r^4\ 0{\cdot}87$$

We may assume here for cast iron,

$$EI = 160{,}000$$

for wrought iron,

$$EI = 2{,}700{,}000$$

and for wood,

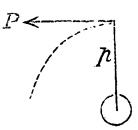
$$EI = 16{,}000$$

If now we consider the general case of an axle submitted both to flexion and to torsion, the expressions for the two efforts which have been already obtained must be added together, and we will find therefore:

$$EI = \frac{V\Sigma M}{\mu} + \frac{M'r}{\mu'}$$

The moment M' is known by considering the horse power transmitted and the number of revolutions of the axle, for the work for one revolution is:

$$P2\pi p$$



and if n be the number of turns per minute, the work per 1′ is:

$$nP2\pi p$$

and in one second:

$$\frac{nP2\pi p}{60}$$

which is equal to the horse power, whence:

$$\frac{nP\pi p}{30} = 75N$$

or,

$$M' = \frac{2250\ N}{\pi n} = Pp$$

In the case considered, then, if we assume the axis to be hollow and $V \stackrel{1}{=} 0{\cdot}6\ r$, we have:

$$EI = \frac{Ql}{0.87\ \pi r^3} + \frac{2Pp}{0.87\ \pi r^3}$$

whence:

$$r = \sqrt[3]{\frac{Ql+2Pp}{0.87\ \pi EI}}$$

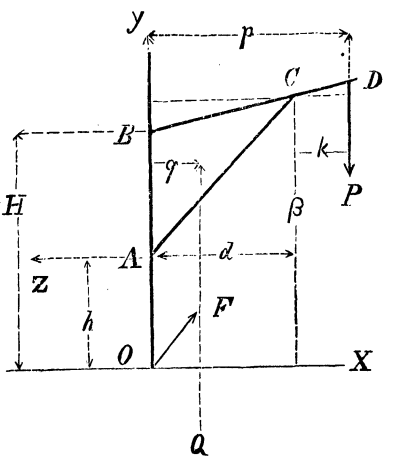
For illustration, assume a hollow cast iron axle of a water wheel, transmitting 50 horse power and making 15 turns per minute, and let $l = 0^m.6$. Then $Ql = 2{,}700$; $Pp = 2{,}387.3$; $EI = 200{,}000$; whence:

$$r = \sqrt[3]{\frac{7{,}474.6}{546{,}889.7}} = 0.239$$

From this dimension the approximate weight of the axle itself could be calculated, and the estimation could be remade taking that weight into consideration.

CRANE.

To determine the dimensions of the pieces of a crane or similar frame, we must first estimate the reactions that strain it at different points. Let X and Y be the horizontal and vertical components of F, and Z be the reaction at A. Z is horizontal, since the frame has no tendency to slide vertically,



and let Q be the weight of the crane. Applying the conditions of equilibrium for forces and moments, we have:

$$X+Z=0; \quad Y-P-Q=0; \quad Pp+Qq-Zh=0;$$

whence,

$$Z=\frac{Pp+Qq}{h} \quad y=P+Q; \quad X=-Z.$$

Let us also determine the particular reactions on the different pieces at the points A, B, and C. Let Ax and Ay be the horizontal and vertical components of the reaction at A in the direction of the strut tending to compress it. Let Bx and By be the components of the reaction at B on the arm of

the crane. Let $-Cx$ and $-Cy$ be the reaction at C of the arm on the strut, while $+Cx$ and Cy are the equal and contrary ones of the strut on the arm at the same point. The equilibrium of the strut gives (neglecting its weight):

$$Ax-Cx = 0, \text{ and } Ay-Cy = 0;$$

also, calling a and b the coordinates of the point C'

$$Ax\,(b-h)-Aya = 0$$

In this last equation the moments are taken about the point C. If we now consider the equilibrium of arm, we see first that the reactions at C disappear as being equal and contrary, and neglecting the weight of the arm, we have:

$$Ax+Bx = 0; \quad Ay+By-P = 0$$

and from moments:

$$Ax\,(b-h)+Bx\,(b-H)-Aya-Bya-Pk = 0$$

or, since $Ax\,(b-h) = Aya$:

$$Bx\,(b-h)-Bya-Pk = 0$$

or, again replacing Bx and By by $-Ax$ and $P-Ay$ we have:

$$Ax\,(b-h)-Aya+P\,(a+k) = 0$$

From these six equations, we may determine Ax, Ay, Bx, By, Cx and Cy.

Taking now the two components, Ax, Ay, and projecting them on the direction AC, we obtain the force which tends to compress the strut, from which its cross section is easily deduced.

Then projecting the components Bx, By, on the direction BC, we likewise obtain the force which

tends to elongate the part BC. The part CD, however, resists a cross strain, for which we must have:

$$EI = \frac{V\,Pk}{\mu}$$

To calculate the section of the upright, we may consider the part OA as fixed at A, and apply the formula for a beam submitted to compression and a cross strain:

$$EI = \frac{V\Sigma M}{\mu} - \frac{T}{\omega}$$

in which

$$\Sigma M = Xh, \text{ and } T = Y, \text{ or } EI = \frac{VXh}{\mu} - \frac{Y}{\omega}$$

The part AB can be considered in a similar manner. Its formula will be:

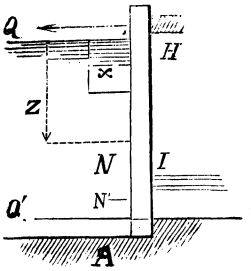
$$EI = \frac{VBx(H-h)}{\mu} - \frac{By}{\omega}$$

In construction, the force P is taken, 10 the force that the machine may be likely to support.

UPRIGHT IN A WOODEN DAM, OR LOCK GATE.

This is a case of beam supported at both ends with a pressure not uniformly distributed over it.

Let the length of the upright be l and its breadth λ, call the head of water on the one side H and on the other h. Let M be any point of the neutral fibre situated at x below the upper level. The pressure at that point is $Px\lambda dx$. P being weight of



unit of volume of water. The total pressure is then on the one side $\frac{1}{2}P\lambda H^2$, and on the other $\frac{1}{2}P\lambda h^2$. Let these pressures be p and p'; then will we have for equilibrium among parallel forces:

$$p-p'-Q-Q' = 0$$

whence,

$$Q+Q' = \tfrac{1}{2}P\lambda\ H^2-h^2)$$

and if we consider the moments of the same forces with reference to A, we shall have, for the moment of an elementary pressure $P\lambda xdx$, the expression $P\lambda xdx(H-x)$; wherefore the whole moment is $\frac{1}{6}P\lambda H^3$ on the one side, and in like manner $\frac{1}{6}P\lambda h^3$ on the other. The moment of Q, moreover, is Ql, and that of Q' is zero. We shall then have for equilibrium:

$$Ql+\tfrac{1}{6}P\lambda h^3 = \tfrac{1}{6}P\lambda H^3$$

or,

$$Q = \frac{P\lambda(H^3-h^3)}{6l}$$

and,

$$Q' = \tfrac{1}{2}P\lambda\left[(H^2-h^2)-\frac{(H^3-h^3)}{3l}\right]$$

Having now found the forces Q, Q' and the expressions for the pressure of water, let us determine ΣM for any point, such as N, at a distance z below the upper level. The elementary moment of the pressure is:

$$P\lambda xdx(z-x)$$

and the entire is :

$$\int_0^z P\lambda x dx(z-x) = \tfrac{1}{6}P\lambda z^3$$

The moment of Q is $Q(z+l-H)$, therefore we may write:

$$\Sigma M = (a)\,Q\,(z+l-H)-\tfrac{1}{6}\,P\lambda z^3$$

If we annul the differential coefficient of this in order to obtain the maximum of ΣM, we shall find;

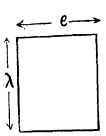
$$0 = Q-\tfrac{1}{2}\,P\lambda z^2$$

whence,

$$z = \sqrt{\frac{2Q}{P\lambda}} = \sqrt{\frac{H^3\,h^3}{3l}}$$

If this value of z refers to a point above the lower level, that is, if $z < H-h$, then by placing it in eq. (a) we obtain the value of ΣM, which is required for the general eq. $EI = \dfrac{V\Sigma M}{\mu}$

If we suppose then that the section of the upright



be rectangular, and that e be its thickness in the direction of the thrust, we shall have :

$$V = \tfrac{1}{2}\,e$$

and,

$$\mu = \tfrac{1}{12}\,\lambda e^3$$

whence,

$$EI = \frac{6\mu}{\lambda e^2}$$

from which a value for e can be obtained when EI is assumed. If, however, this value of z for a maximum be found to relate to a point below the lower level, that is to say, if z be found $z > H-h$, then the calculation must be remade, taking into account the moment of the pressure in the lower level. Suppose that in fact some point, such as N', be the point of maximum strain, then an elementary pressure below being $P\lambda dx\Big(x-(H-h)\Big)$ its moment with reference to the point N' is:

$$P\lambda dx\Big(x-(H-h)\Big)(z-x)$$

and that integrated between $H-h$ and z will give the whole moment which must be inserted together, with the other two already found, in eq. (a). This then becomes:

$$(b)\ \Sigma M = Q(z+l-H)+\tfrac{1}{2}P\lambda z(H-h)^2-\tfrac{1}{2}P\lambda z^2(H-h)-\tfrac{1}{6}P\lambda(H-h)^3$$

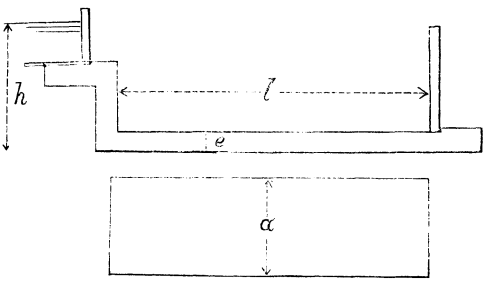
an expression which becomes maximum for:

$$z = \tfrac{1}{2}(H-h)+\frac{Q}{P\lambda(H-h)}$$

Replacing this value of z in eq. (b), we obtain the expression for ΣM, which we have to treat in the new application of eq. $EI = \dfrac{V\Sigma M}{\mu}$

BOTTOM OF LOCK, MASONRY APRON, &c.

This volume of masonry, *ael*, is subjected, when the lock is empty, to a pressure p' per unit of length,



which is the difference between its weight and the upward pressure of water which may filter below it and whose thrust is *pah*. To determine the dimension of this solid, it is usually considered as a beam fastened at its two ends and uniformly loaded; we will then have for this case:

$$\Sigma MQ = \tfrac{1}{12}\, p'l^2$$

or,

$$EI = \tfrac{1}{12}\,\frac{Vp'l^2}{\mu}$$

But,

$$V = \tfrac{1}{2}\, e$$

and,

$$p' = a\,(ph - Pe)$$

in which $p = 1000^{\text{kg}}$ and P is the weight of a cubic millimetre of the masonry. Also, $\mu = \tfrac{1}{12}\, ae^3$. Substituting these values, we have:

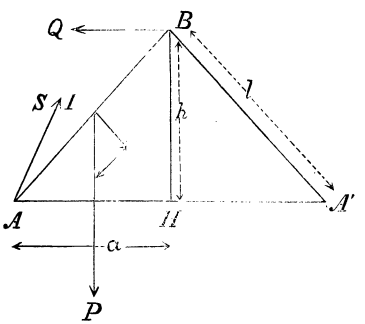
$$EI = \tfrac{1}{2}\,\frac{ea\,(ph - Pe)l^2}{ae^3} = \frac{(ph - Pe)\,l^2}{2e^2}$$

For EI we may assume $0^{kg}.037$ per millimetre,[2] which is $\frac{1}{10}$ of the load which would crush mortar. The equation which gives e is of the second degree, but the negative root will always be rejected.

TRUSSES.

Truss composed of two rafters and a tie beam.

Let P be the weight of roofing borne, uniformly distributed on one rafter. Let the reactions at the



extremities of the rafter be S and Q; Q being horizontal, decompose S into Sx and Sy. We shall have:

$$Sx + Q = 0$$

$$Sy + P = 0$$

and,

$$P\tfrac{1}{2}a - Qh = 0$$

whence,

$$Q = P\frac{a}{2h}$$

$$Sx = -\frac{Pa}{2h}$$

$$Sy = -P$$

Let p be the weight of roofing per square metre, and let e be the distance apart of two consecutive rafters, we shall then have:

$$P = pel$$

The rafter may now be considered as a beam supported at A and B and uniformly loaded. The weight it bears per unit of length is pe and the component of this weight perpendicular to the beam is:

$$pe\frac{a}{l}$$

Besides this uniform load it is subjected to the components of P and Q longitudinally; or to

$$T = P\frac{h}{l} + Q\frac{a}{l}$$

or,

$$T = peh + pe\frac{a^2}{2h}$$

The maximum moment of flexure is at the point I. The component of Q perpendicular to the beam is one-half the component of P in the same direction; therefore, we have:

$$\Sigma MQ = \tfrac{1}{8}\, pel^2$$

If the section be a square and x its side, we shal have the eq.:

$$EI = \frac{\Sigma MQ\, V}{\mu} - \frac{T}{\omega}$$

$$V = \tfrac{1}{2}\, x$$

$$\mu = \tfrac{1}{12}\, x^4$$

$$\omega = x^2$$

whence :

$$EI = \tfrac{3}{4}\frac{pel^2}{x^3} - \frac{pe}{x^2}\left(h + \frac{a^2}{2h}\right) \quad (a)$$

The second term of the second number is generally omitted in order to solve the eq., and we first take :

$$EI = \tfrac{3}{4}\frac{pel^2}{x^3}$$

or,

$$x = \sqrt[3]{\tfrac{3}{4}\frac{pel^2}{EI}} \quad (b)$$

in which $EI = 6{,}000{,}000$ for iron, and 70,000 for oak. If we desire a greater approximation for x, the value for x in eq. (b) may be taken and substituted in eq. (a), from which a new value of EI is obtained which may be substituted in eq. (b).

The tie is solicited by its weight, and if besides it supports a flooring, it is also submitted to an uniformly distributed pressure. It supports also the horizontal component of the rafter which is a tension at $A = Sx = \frac{Pa}{2h}$ as also an equal one at A'. If the tie be simply a wrought iron rod, its weight may be neglected and its section supposed circular, calculated from the formula $\frac{Pa}{2h} = EI\pi r^2$. If, however, the tie be of wood, and moreover support a flooring weighing q per linear metre of the span, we must employ the formula $EI\frac{\Sigma MQ\,V}{\mu} - \frac{T}{\omega}$ in which

$$\Sigma MQ = \tfrac{1}{2}\,qa^2$$

$$T = \frac{pa}{2h}$$

and,

$$\omega = x^2$$

then,

$$V = \tfrac{1}{2}x$$

and,

$$\mu = \tfrac{1}{12} x^4$$

consequently,

$$EI = \frac{3qa^2}{x^3} - \frac{Pa}{2hx^2}$$

whence x may be determined by approximations as in the case of the rafter. The value for EI would reduce to the second degree of x if the tie bore no flooring, for then calling P the weight of a cubic metre of the wood forming the tie, we should have:

$$q = Px^2$$

and consequently,

$$EI = \frac{3Pa^2}{x} - \frac{Pa}{2hx^2}$$

But in the general case if the section were a rectangle, its dimensions might still be called x and kx, whence,

$$V = \tfrac{1}{2} kx$$

$$\mu = \tfrac{1}{12} xk^3x^3$$

and the equation would still present the same form as at first. If the truss had no tie, the force Sx would represent with a contrary sign the thrust exercised by the rafter on the wall. Suppose, now, that the truss present in its composition, a king-post BH; this last we know is subjected to a tension by the sagging of the tie. Let us then, for the moment, consider the tie as a beam placed upon three supports, viz: the two walls and the stirrup of the king-post. In such case the reactions on the extremities are $\frac{3}{8} qa$ and that on the middle support is $\frac{5}{4} qa$. The king-post is then extended by a force $= \frac{5}{4} qa$, and each rafter can be considered as subjected at its upper end to a vertical force $= \frac{5}{8} qa$. In this case the equations of equilibrium of the rafter become:

$$Sx + Q = 0$$

$$Sy + P + \tfrac{5}{8} qa = 0$$

and,

$$P\tfrac{1}{2}a+\tfrac{5}{8}qa^2-Qh=0$$

whence :

$$Q=\frac{Pa}{2h}+\frac{5qa^2}{8h}$$

$$Sy=-(P+\tfrac{5}{8}qa)$$

and,

$$Sx=-Q$$

wherefrom we should find :

$$T=pe\left(h+\frac{a^2}{2h}\right)+\frac{5qa}{8l}\left(h+\frac{a^2}{h}\right)$$

which value could be used instead of

$$T=pe\left(h+\frac{a^2}{h}\right)$$

to determine as before, the dimensions of the rafter. The dimensions of the tie would have to be obtained by considering it as a solid supported at three points ; we should find :

$$\Sigma MQ=\tfrac{1}{8}qa^2$$

for the maximum moment of flexure, and

$$T=\frac{Pa}{2h}+\frac{5qa^2}{2h}$$

These values substituted in

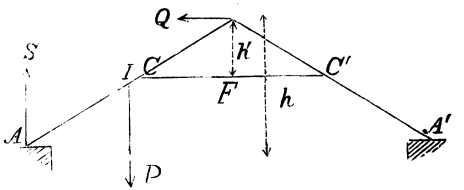
$$EI=\frac{\Sigma MQV}{\mu}-\frac{T}{\omega}$$

and reduced as has been already described, will give the required side of square or rectangle.

The section of the king-post will be given approximately by the condition that it supports a longitudinal effort equal to $\frac{5}{4}qa$, whence :

$$x=\sqrt{\frac{5qa}{4EI}}$$

Suppose now, a truss tied at some points, C, C' in such manner that the tension F be absolutely equal to the reaction Q of the one rafter on the other, so



that the walls in fact support only a vertical pressure. The equilibrium of the rafter gives, then :

$$F = Q \quad S = P \quad P\tfrac{1}{2}a + F(h - h') - Qh = 0$$

From this last, we have :

$$P\tfrac{1}{2}a - Qh' = 0$$

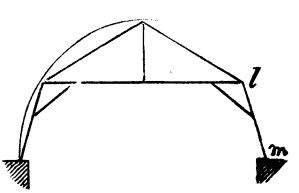
or,

$$Q = \frac{Pa}{2h^1}$$

The rafter is then considered as placed upon two supports A and B charged with a load $pe\frac{a}{l}$ uniformly distributed along its length and perpendicular to its axis, and subjected besides to a cross force $F\frac{h}{l}$ and to three longitudinal ones $F\frac{a}{l}$, $Q\frac{a}{l}$ and peh

The point of maximum moment of flexure can be easily determined and the dimensions of the rafter calculated as already described.

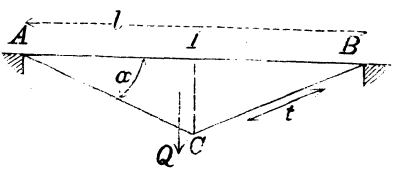
A mansard roof is in composition nothing more than a truss similar to the foregoing, supported on legs lm (jambes de force). These last being but



slightly inclined are calculated solely for the vertical pressure which they sustain. Each is assumed to support half the weight of the truss and the formula to be applied is $\omega EI = W$. The little ties and braces accessory in a frame, but which are not necessary to equilibrium, cannot be calculated; their dimensions are empiric.

BEAM TRUSSED BY TIES AND A VERTICAL SUPPORTING STRUT BELOW.

This piece must be considered as resting upon three supports. The reaction of the middle support is $\frac{5}{8}pl$, p being the weight per metre uniformly



loaded on the span. This reaction is equal to the pressure Q supported by the strut, or $Q = \frac{5}{8}pl$, and let t be the tension of a tie. If now we consider the equilibrium of C, we have:

$$Q - 2t \sin.\alpha = 0,$$

whence,

$$t = \frac{16 \sin.\alpha}{5pl}$$

and $t \cos.\alpha$ is the tension in the direction AB. Therefore for the beam AB we will apply as before the formula,

$$EI = \frac{\Sigma MQv}{\mu} - \frac{T}{\omega}$$

in which,

$$\Sigma MQ = \tfrac{1}{8}\, p\left(\frac{l}{2}\right)^2 = \frac{pl^2}{32}$$

and, as

$$\frac{T}{\omega} = -\frac{t\cos.a}{\omega}$$

we have,

$$EI = \frac{vpl^2}{32\mu} + \frac{t\cos.a}{\omega}$$

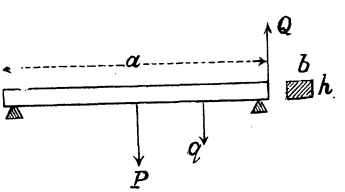
or, replacing t by its value

$$EI = \frac{vpl^2}{32\mu} + \frac{5pl\cot.a}{16\omega}$$

The length of the strut IC is generally $\tfrac{1}{6}l$, therefore, $a = 18^\circ\, 26'$.

DIMENSIONS FOR TIMBER BRIDGES.

If a beam supporting part of a roadway form a whole bay, it is considered simply as resting upon supports at its ends and subjected to one constant uniform load p, to another p' for security fixed by law, and to a weight P which may be applied at the



middle representing the passage of a locomotive or other weighty vehicle. The reaction Q is then :

$$Q = \tfrac{1}{2}\left((p+p')a + P\right)$$

and the force

$$q = \tfrac{1}{2}(p+p')a$$

The moment of Q is:

$$\tfrac{1}{2}\Big((p+p')a+P\Big)\frac{a}{2}$$

and that of q is:

$$\tfrac{1}{2}(p+p')\frac{a}{4}$$

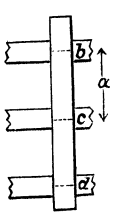
their difference is:

$$\Sigma M = \tfrac{1}{8}(p+p')a^2+\tfrac{1}{4}Pa$$

replacing which in the formula, we have:

$$EI = \frac{6}{bh^2}\Big(\tfrac{1}{8}(p+p')a^2+\tfrac{1}{4}Pa\Big)$$

in which $EI = 600{,}000$ for oak, and $800{,}000$ for pine. Having thus deduced the cross section hb, the weight of the beam may be calculated and introduced as a consideration for a closer approximation. When the same beam forms more than one bay it is considered as supported at various points; the reactions must be determined and the maximum moment obtained for which the dimensions must be adapted as will be described. The roadway bearers which cross the main beams are also in the condition of prisms supported at several points and loaded at their middle as well as uniformly loaded. Instead of calculating them from this view, however, it is



sufficient to considerany part, as *bc*, *cd* as separate and independent, then to calculate its dimensions first as if it were simply resting on supports, and secondly as if it were fixed at the ends, and take the mean of the two results. Thus in the first calculation we should have :

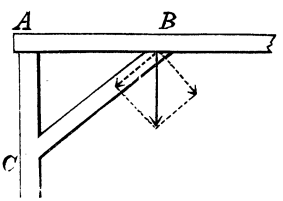
$$\Sigma M = \tfrac{1}{8}\, pa^2 + \tfrac{1}{4} Pa$$

and in the second :

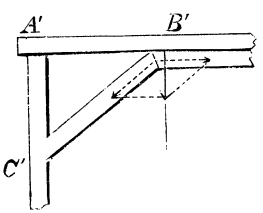
$$\Sigma M = \tfrac{1}{12}\, pa^2 + \tfrac{1}{8} Pa$$

The half sum of these moments would be the expression to replace for ΣM in the formula.

To calculate struts, the beam is supposed supported at the points A, B ; and the reaction at B



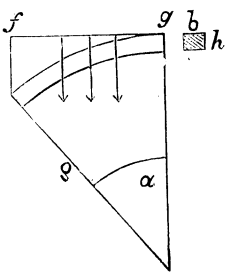
is determined, and, for a strut CB without straining beam, the component of the reaction in direction of the strut is taken and the cross section calculated for compression from that, assuming 60 to 80 kil. per centimetre square as the allowance. In case of



the strut $B'C'$ with straining beam the reaction at B' when determined is decomposed in the two directions of the strut and straining beam, and these components are used to determine the respective cross sections by the formula $T = EI\omega$ in which always $EI = 600{,}000$ to $800{,}000$.

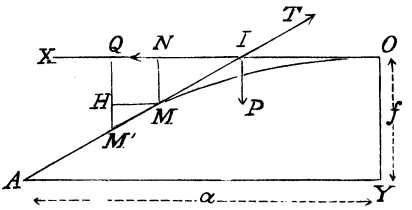
DIMENSIONS OF CURVED BEAMS.

Without going at present into the minute consideration of the resistance of curved pieces which will be hereafter discussed, we may, in practice, consider that the weight of roadway, &c., as uniformly distributed on the projection fg of any



portion of arc and apply the following formula $bh^2 = \frac{P}{300{,}000}\left(\frac{h}{a} + \frac{\varrho a^3}{4}\right)$ in which P is the whole weight of the bridge on the arc ϱa. However, as such a beam is supposed to resist by compression alone, its strain can be easily determined, and therefrom its cross section, if its curve be supposed a parabola, as follows:

Take the origin at O and from any point M draw the tangent. It cuts OX at the middle of ON.



The weight of the bridge supposed to be uniformly distributed on the projection ON is P acting at I.

Draw $HM = dx$ and $HM' = dy$. Now the arc MO is in equlibrium under the action of T, Q, and P. Wherefore, since T for instance, then is the resultant of Q and P, these three forces must be proportional to the sides of the triangle MIN, or what is the same, to top of the triangle $MM'H$. Therefore:

$$\frac{M'H}{MH} = \frac{P}{Q} = \frac{px}{Q} = \frac{dy}{dx}$$

Integrating and observing that for $x = 0$, $y = 0$ the equation of the curve becomes $y = \frac{p}{2Q}x^2$ This being true for all values, we must have:

$$f = \frac{p}{2Q}a^2$$

whence,

$$Q = \frac{pa^2}{2f}$$

Moreover,

$$T = \sqrt{P^2 + Q^2} = p\sqrt{x^2 + \frac{a^4}{4f^2}}$$

and as the maximum of T corresponds to $x = a$, it becomes:

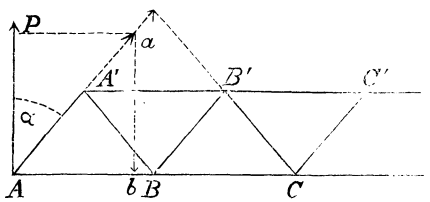
$$T = pa\sqrt{1 + \frac{a^2}{4f^2}}$$

This tension divided by the cross section being the value EI, the dimensions are given when EI is assumed. If we call $EI = 600{,}000$, that is to say, if we allow $0^{kg}.6$ per millimetre square, then:

$$\frac{T}{bh} = 0.6$$

LATTICE TRUSS.

Let us consider a simple lattice truss formed by two horizontal pieces, connected by inclined ties and braces as in the figure. Let it be supposed subjected



to a force $2P$ applied at the middle, so that the reaction at one extremity is a vertical force P. To determine the strains on the different members; commencing at the point A it is evident that there is equilibrium between P, Aa and Ab, whence from the parallelogram:

$$Aa = \frac{P}{\cos.\alpha}$$

and,

$$Ab = P \text{ tang.}\alpha$$

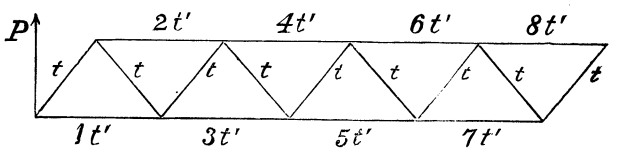
Considering the point A' we see again that the tensions in the direction $A'a$ and $A'B$ are equal, and therefore equal to $\frac{P}{\cos.\alpha}$, and also that the tension in the direction $A'B'$ is equal to $\frac{2P}{\cos.\alpha}\sin.\alpha = 2P \text{ tang.}\alpha$. The equilibrium of B shows, by projecting the forces vertically, that the pressures of $A'B$ and BB' are equal; and by projecting horizontally, that the tension of BC is equal to

$$P \text{ tang.}\alpha + 2\frac{P}{\cos.\alpha}\sin.\alpha = 3P\text{tang.}\alpha$$

Operating in the same way for B', it is found that the pressure of $B'C$ is equal to that of BB', and that the tension of $B'C'$ is equal to

$$2\,P\,\text{tang.}\alpha + 2\frac{P}{\cos.\alpha}\sin.\alpha = 4\,P\,\text{tang.}\alpha$$

Continuing thus, it appears that all the inclined members support the same pressure equal to $\frac{P}{\cos.\alpha}$; that the horizontal lower parts support tensions successively expressed by P tang.α, $3\,P$ tang.α, $5\,P$ tang.α, $7\,P$ tang.α, &c., and the horizontal upper parts, tensions $2\,P$ tang.α, $4\,P$ tang.α, $6\,P$ tang.α, $8\,P$ tang.α, and so forth on to the middle of the bay. The other half of the truss is in the same conditions.



$$t = \frac{P}{\cos.\alpha},\quad t' = P\,\text{tang.}\,\alpha$$

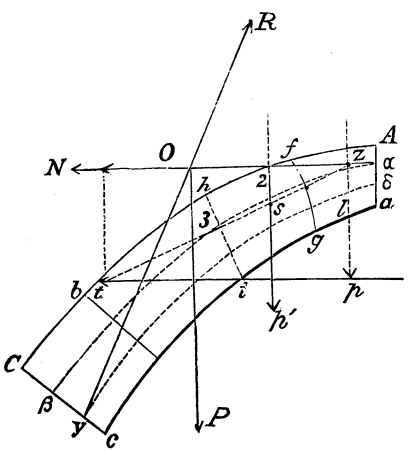
If the number of the lower horizontal members be uneven, the middle span supports the greatest tension, and calling $2n+1$ the number of them all, the maximum tension is expressed by $(2n+1)\,P$ tang.α. If the number of lower members be even, then the upper middle span is most strained, and if $2n$ be the number of spars, its tension is $2n\,P$ tang.α. These considerations might be carried farther, but to calculate generally the section of an ordinary lattice truss, such as Town's, any one truss may be considered as a beam on two supports loaded uniformly, and also liable to the weight of two locomotives at the middle. But as the empty spaces of the lattice

are about two-thirds of the total volume, the coefficient E must be reduced, so that EI may be only assumed 200,000. Calling then p the weight of the bridge per lineal metre; P, the overweight at the middle of bay; a, the length of it; b, the breadth of a truss, and h its height, we may apply with perfect safety the formula

$$200{,}000 = \frac{6}{6h^2}\left(\tfrac{1}{8}pa^2 + \tfrac{1}{4}Pa\right)$$

DIMENSIONS OF VOUSSOIRS.

Let $ACca$ be a portion of a stone arch. The simplest method of determining its dimension is a graphic one. Lay off the curves aB, $y\delta$, passing



each through points distant one-third of the thickness from the outer and inner edges. Within this space $aBy\delta$ must the resultants of all the pressures on the various voussoirs intersect the joints, for it

can be shown that if they do not do so, there will be a negative pressure on some part of the joint, or in other words, no bearing at all. Draw then through a the horizontal reaction aN at the crown of the arch, and let OP be the resultant of all the pressures on the half arch, including its own weight. The reaction of the abutment, if the arch be in equilibrium, must pass through O and between B and y; draw it from y to O. Now P being known entirely, ON and OR are known, that is to say the reaction of the crown and the reaction of the abutment. Those forces are pressures on the joints Aa and Cc and may serve to determine their dimensions, supposing always that after the graphic construction be described, it is found that yOR does fall within By. Consider then now the upper voussoir: draw p the resultant of the external pressures on it, no matter what their direction; if the external pressure is a weight of masonry, this resultant may be considered vertical; if of water, it is normal to the extrados, &c.; at all events, the voussoir is in equlibrium under the action of ON, of p, and of the reaction of voussoir 2; draw then that reaction as the resultant of p and ON from the point z. This must intersect the joint fg within the lines aB, $y\delta$, and and also for a good establishment be normal to fg. If it be so, the construction so far is satisfactory, and the resultant is the pressure which the joint fg must resist, and serves therefore to determine its dimensions. The voussoir 2 is in equilibrium under the action of p'', of the resultant zst, and of the reaction of voussoir 3. The direction and intensity of this reaction can therefore be determined, and suffice for the calculation of the area of the joint hi, supposing always that the direction falls within δB, δy and is approximately normal to the joint. Thus may all the pressures be determined graphically, and the last reaction yOR must in turn fall within the limited space, and be perpendicular to Cc. Ultimately these dimensious are calculated on the supposition that EI for cut stone is 50,000, and for

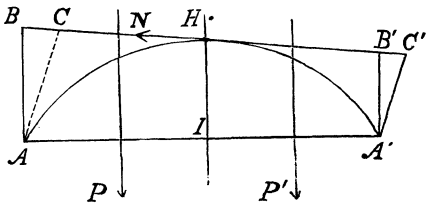
rubble 10,000. The weight of masonry may be evaluated at $2,200^{kg}$ per cubic metre, and the over charge of arches at $1,320^{kg}$. To make the first diagram for this graphic solution, the thickness may be taken $e = 0^{m}.325 + 0.0347l$, in which l is the span; the curve of the arch then being chosen and the figure drawn with that condition, it remains to construct the various lines and resultants and see whether the conditions of stability are fulfilled, if so, the dimensions follow; if not, the arch must be made thicker. If it seems already too thick, it may be reduced.

METALLIC BRIDGES.

If the trusses are curved, the arch is made of cast iron, if the trusses are not curved, the beams are of wrought or sheet iron. A cast iron arched truss is subjected only to compression; its voussoirs can be calculated precisely like those of a masonry bridge. But in that case we assumed the reaction at the crown to be horizontal, and as in metallic trusses, the inequalities of load are more important than in a stone arch, it is usual to determine the reaction at the crown supposing that the two halves do not bear the same weight, or that while one supports its own weight the other supports that and an extra load beside of 400^{kg} per square millimetre.

To determine graphically the direction of the reaction at H, suppose P and P' the weights supported by the two halves of the arch. Let H be the point of application of the reaction and BC' its direction. From the springing points of the curve A, A', draw the perpendiculars AC $A'C'$ to this direction and raise the vertical AB, $A'B'$; let p, p'

be the distance of the forces PP' from A and A'. Then for the equlibrium of the first half of the arch,



we must have the condition $NAC = Pp$ and for the second half $NA'C' = P'p$, whence:

$$\frac{AC}{A'C'} = \frac{Pp}{P'p'}$$

But the triangles ABC $A'B'C'$ being similar, the first member of this last equation can be replaced by $\frac{AB}{A'B'}$ whence:

$$\frac{AB}{A'B'} = \frac{Pp}{P'p}$$

or,

$$\frac{AB}{AB+A'B'} = \frac{Pp}{Pp+P'p'}$$

But calling $IH = H$ we have:

$$AB+A'B' = 2H$$

and consequently:

$$AB = 2H \frac{Pp}{Pp+P'p'}$$

which gives the direction of the force N, and therefore the distance AC, and consequently the inten-

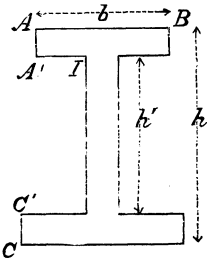
sity of the thrust N. The reaction thus obtained can then be used in the graphic solution given for determining the pressures on all the voussoirs. These pressures being obtained in intensity and direction, the voussoirs may be calculated as cast iron prisms subjected to forces which both tend to bend and crush them. For cast iron, the tension per millimetre should not be allowed greater than 2^{kg}, while the pressure may be 10^{kg} As for arches formed of continuous pieces they are treated like curved wooden beams. A hollow wrought iron beam such as the Britannia bridge, is calculated as a prism, supported and loaded with a weight uniformly distributed and a surcharge applied at the middle of bay. If the beam be on two supports and if p represent the weight per millimetre uniformly distributed and P the surcharge, a, the length of bay, h, the height of the beam or tube and μ the moment of inertia of its right section referred to, the horizontal passing through its centre of gravity, then:

$$EI = \frac{\frac{1}{2}h}{\mu}\left(\tfrac{1}{8}pa^2 + \tfrac{1}{4}Pa\right)$$

The tension allowed for the iron of this sort of beam is 12 kilogramms per square millimetre, or $EI = 12{,}000{,}000$. If the tube be supported at several points it is treated like an ordinary beam in same case; the reactions are determined, the maximum moment of flexure found, and the area sufficient for resistance calculated as usual.

In metallic arched bridges the arcs are ordinarily cast iron of double T cross section, but sometimes the condition is imposed that their resistance be the same in all sections, at least in those which are near the crown. This can be effected by varying the distance between the interior edges of the flanges; if, for instance, h designates the distance between the summits A and C, b the distance between the points A and B, h' the distance between A' and C'

and b' double the distance AI, we can cause h' to to vary in such manner that the beam may be



approximately a solid of equal resistance. Let p be the uniform load, a, the length of the bay; each point of support exerts a reaction $\frac{1}{2}pa$, the moment of the forces which act upon bay from the point whose abscissa is x to the other end is:

$$\Sigma M = p(a-x)\,\tfrac{1}{2}(a-x)-\tfrac{1}{2}pa(a-x)$$

or,

$$\Sigma M = -p\,(ax-x^2)$$

The moment of inertia of the section of the beam is:

$$\mu = \tfrac{1}{12}(bh^3-b'h'^3)$$

Finally, the ordinate of the extreme fibre is:

$$V = \tfrac{1}{2}h$$

whence,

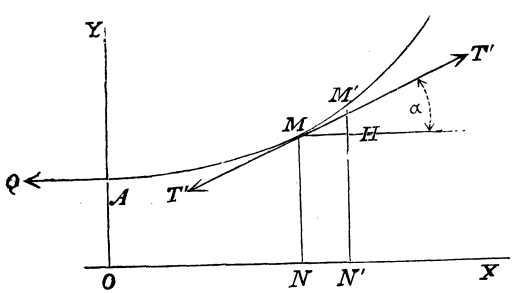
$$El = \frac{\Sigma MV}{\mu} = \frac{6p\,(ax-x^2)}{bh^3-b'h'^3}$$

a formula which gives values of h' corresponding to values of x. But the thickness AA' must not be less than the thickness of the web, so that if a certain value of h' made AA' we should cease increasing h' but obtain the economy by decreasing h.

As an illustration, $p = 1600$; $h = 0^m.60$; $b = 0^m.28$; $b' = 0.26$; $EI = 6{,}000{,}000$; we would find for h' at the crown, that is, when $x = \frac{1}{2}a$, $h' = 0^m.52$, which gives $0^m.08$ for the thickness of the two flanges together, or 0.04 for each. We would find that at $x = 1^m.44$ the thickness of each flange is reduced to 0.02 which is the thickness of the web; then h and h' must both be diminished by the same quantity without diminishing the thickness of the web till h becomes $0^m.40$ which may be its limit fixed beforehand.

SUSPENDED BRIDGES.

The first study in these is to determine the geometric locus of the polygon formed by the points of attachment of the suspending rods to the cable. Each couple of rods corresponding on the two sides of the roadway, can be considered as bearing half the weight of the two roadway spaces between them and the preceding and following couples of rods. If the rods are equidistant, then they all support equal tensions. If they were infinite in number and infinitely near together, each couple would sustain an element of the roadway and any portion of the cable or chain would bear a weight proportional to its horizontal projection. In this hypothesis, it is easy to show that neglecting the weight of the rods



the cable would affect the form of a parabola with vertical axis. Let A be the lowest point of the chain. Here is exerted a horizontal tension which we will call Q. Draw the axes, let M and M' be infinitely near to each other, and let T be the tension of the chain at the point M.

At M the components of T are $T\cos.\alpha$ and $T\sin.\alpha$. Passing from M to M', these components become:

$$T\cos.\alpha + dT\cos.\alpha$$

and,

$$T\sin.\alpha + dT\sin.\alpha$$

Let $2p$ be the weight of bridge per lineal metre, then $2pdx$ will be the weight of an element, and pdx will be the weight of the portion NN' of the bridge sustained by a rod articulated at the middle of the element MM'. This element being in equilibrium under the action of this weight and of the two tensions just considered, we shall have, taking the sum of the horizontal components:

$$(T\cos.\alpha + d'T\cos.\alpha) - T\cos.\alpha = 0$$

or,

$$dT\cos.\alpha = 0$$

whence,

$$T\cos.\alpha = \text{constant} = Q$$

Taking the vertical componets, we shall have:

$$(T\sin.\alpha + dT\sin.\alpha) - T\sin.\alpha - pdx = 0$$

or,

$$dT\sin.\alpha = pdx$$

and putting for T its value $\frac{Q}{\cos.\alpha}$ we have:

$$dQ\text{tang}.\alpha = pdx$$

or,

$$Qdy' = pdx$$

calling y' the angular coefficient of the tangent at M. Integrating and observing that the point A being the lowest, we have:

$$y' = 0. \quad \text{For } x = 0$$

we obtain:

$$Qy' = px$$

or,

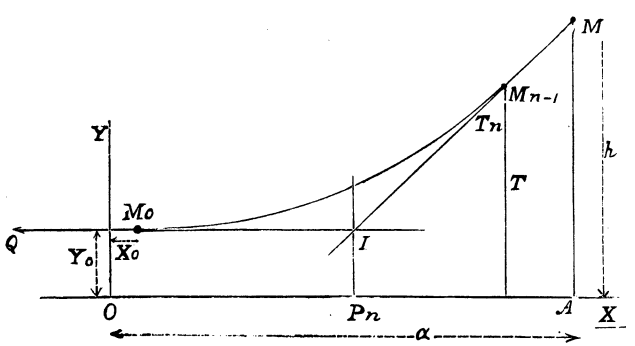
$$dy = \frac{p}{Q} xdx$$

which integrated becomes:

$$y - y_0 = \frac{p}{2Q} x^2$$

equation of a parabola.

Finally, it can be shown that neglecting the weight always of the rods, their points of attachment to the cable are on an arc of parabola, no matter what their number or the regularity of their spacing. It is sufficient to admit that, here without farther demonstration, and to proceed to determine the tensions, assuming the general equation of the curve as $y - y_0 = \frac{P}{2Q}(x^2 - x_0^2)$. The point where



the cable meets the vertical through the end of the bridge Mn is always given; if the chain is symmetric with regard to the lowest point, the co-ordinates of this point are one-half the span a and the height h above the roadway, the axis of x being taken at the roadway level. Substituting these values in the general equation of the curve, we have:

$$h-y_0 = \frac{p}{2Q}(a^2-x_0^2)$$

or,

$$Q = \tfrac{1}{2}p\,\frac{a^2-x_0^2}{h-y_0}$$

in which x_0 and y_0 are the co-ordinates of the lowest articulations. If one articulation or a joint instead of a link of chain came at the lowest point, then would $x_0 = 0$ and the last equation become

$$Q = \tfrac{1}{2}p\,\frac{a^2}{h-y_0}$$

Now to determine the tension of any side of the polygon formed by the chain, let M_{n-1} and M_n be two consecutive summits and T_n be the tension of the side $M_{n-1}\,Mn$. The portion of chain comprised between the points M_0 and M_{n-1} bears the weight of the roadway comprised between the axis of y and the vertical passing through the middle of $M_{n-1}\,M_n$. The value of this weight is,

$$P_n = p\left(x_{n-1}+\frac{x_n-x_{n-1}}{2}\right) = \tfrac{1}{2}p\,(x_{n-1}+x_n)$$

This portion of chain is in equilibrium under the action of its weight and of the forces Q and T_n. Calling a_n the angle of the side $M_{n-1}\,M_n$, we shall have:

$$T_n \cos.a_n - Q = 0$$

and,

$$Tn \sin.a_n = P_n$$

or,

$$T_n = \sqrt{Q^2 + P_n{}^2}$$

This tension augments with P_n; the maximum is then on the last link. To calculate it more easily, we can suppose that the lowest point of the chain is the summit of a parabola, and if we call $b = h - y_0$, we have:

$$Q = \tfrac{1}{2} p \frac{a^2}{b}$$

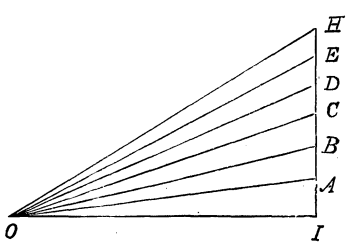
Also,

$$P = pa$$

consequently, calling T the maximum tension, we have:

$$T = pa\sqrt{1 + \frac{a^2}{4b^2}}$$

The horizontal tension and all the others of the chain may also be easily evaluated graphically when the inclination of the last link or side of polygon formed by the cable is known. Upon the indefinite



vertical IH, lay off IA, AB, &c., proportional to the successive weights that the different rods support, that is to the half sum of the two roadway spaces on each side of any couple of rods. Through the point I, draw a horizontal, and through the

extreme point H, draw a right line having the inclination of the last link or polygon side in question. These lines meet at O. Draw OA, OB, &c. The line IH represents the weight borne by all the rods; OI represents the horizontal tension Q, and the lines OA, OB, &c., represent the tensions of the successive sides or links. OH is the last and the maximum. In fact, considering first that portion of chain comprised between the first rod and the last, it is evidently in equilibrium under the action of the horizontal tension Q, the tension T, and the weight P of the roadway; but these three forces being parallel to the three sides of the triangle OIH, are proportional to these sides, and since P is represented by IH it follows that Q is represented by OI and T by OH. Consider then the point of attaching of the last rod; it is in equilibrium under the action of p, of T and of T'. But p and T we admit are represented by DH and O. Therefore T' is represented in intensity and direction by OD which makes, in short, the demonstratur general.

LENGTH OF THE CURVE.

To obtain the weight of the cable, which is an element of the strain upon it, the length must be determined. This could be done by calculting the length of each side of the polygon by means of its co-ordinates, but it simpler to consider the polygon as the parabolic arc circumscribed. Let its equation be $y = \frac{p}{2Q}x^2$ and substituting for Q its value $\frac{1}{2}p\frac{a}{b}$; $y = b\frac{x^2}{a^2}$. If s is the length of the arc to be evaluated, we have:

$$ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

but,

$$\frac{dy}{dx} = 2\,b\,\frac{x}{a^2}$$

whence,

$$ds = dx\sqrt{1+\frac{4\,b^2x^2}{a^4}}$$

In ordinary applications b is small compared to a, and x is at utmost equal to a, therefore $\frac{b^2\,x^2}{a^4}$ is small, and if we add $\frac{4\,b^4x^4}{a^8}$ to the quantities under the radical it will not much affect its value. Doing so, however, that quantity becomes a perfect square; whence,

$$ds = dx\left(1+\frac{2\,b^2x^2}{a^4}\right)$$

Integrating, we shall have:

$$s = x+\tfrac{2}{3}\,\frac{b^2x^3}{a^4}$$

and for $x = a$,

$$s = a+\tfrac{2}{3}\frac{b^2}{a} = a\left(1+\tfrac{2}{3}\frac{b^2}{a^2}\right)$$

This is the length of the parabola from the lowest point to that which corresponds to the end of the roadway. This is moreover the definitive length of the half chain under the action of the weight which it supports; this length must surpass the original length s_0 by the extension due to the tension. Let θ be a mean between the maximum tension and the horizontal tension; Q, the extension of iron per millimetre of length, being $0^m.00005$ for a tension of one kilogramme per millimetre square, we shall have, calling ω the section of the chain in square millimetres:

$$S-S_0 = S_0\,\frac{\theta}{\omega}\,0^m.00005$$

whence,

$$S_0 = \frac{S}{1+\frac{\theta}{\omega}0.00005}$$

The length s may vary with the temperature also. The coefficient of dilatation of iron is 0.0000122, so that passing from 0 to the temperature t the length s becomes $S' = S(1+0.0000122t)$, whence results are increase of versin, which it is necessary to know how to calculate. Now if in the formula

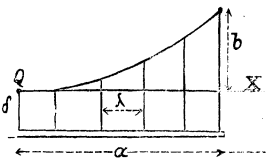
$$S = a\left(1+\tfrac{2}{3}\frac{b^2}{a^2}\right)$$

we make S and b variable and differentiate, we find :

$$ds = \tfrac{4}{3}\frac{b}{a}db$$

For ordinary dilatations, we may suppose the entire actual increase of length of chain and versin also to be proportional to ds and to db. Having given then the increase Δs in any circumstance of ordinary temperature, we may write, $\Delta b = \frac{3}{4}\frac{a}{b}\Delta s$. The section of the suspending rods can easily be calculated, knowing the weight of roadway. The section of the chain is determined from the tension it supports at any point. This tension is first determined using alone the weight of roadway and surcharge. A section being therefrom calculated, the weight of the chain of this section and of the estimated length, must be introduced into a new similar calculation for the determination of another section which may be considered sufficiently approximate. The roadway bearers which the suspending rods support can be considered as prisms on two supports loaded uniformly. If the weight of the rods be added in considering the tension on the chains, it becomes necessary to calculate the sum of their

lengths. Considering then the chain as a parabola and the supporting rods as ordinates spaced apart a



distance $= \lambda$ we can write from the equation $y = \frac{bx^2}{a^2}$ the successive lengths of the rods counting from the lowest point of the chain, as follows:

$$\frac{b\lambda^2}{a^2};\ \frac{b4\lambda^2}{a^2};\ \frac{b9\lambda^2}{a^2}\ \cdots\cdots\ \frac{bn^2\lambda^2}{a^2}$$

the sum Σ of these lengths is then:

$$\Sigma = \frac{b\lambda^2}{a^2}(1+4+9+\cdots\cdots+n^2);$$

but the sum of the squares of the consecutive numbers from 1 to n, is

$$\frac{n(n+1)(2n+1)}{6}$$

therefore,

$$\Sigma = \frac{b\lambda^2}{a^2}\ \frac{n(n+1)(2n+1)}{6}$$

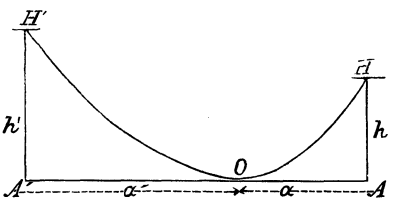
this expression may be simplified by remarking that $a = (n+1)\lambda$ or, $\frac{\lambda}{a} = \frac{1}{n+1}$; putting for $\frac{\lambda}{a}$ this value, and reducing, we obtain,

$$\Sigma = b\ \frac{n}{6}\ \frac{(2n+1)}{(n+1)} = \tfrac{1}{3}bn\ \frac{2n+1}{2n+2}$$

or for a sufficient approximation,

$$\Sigma = \tfrac{1}{3}bn$$

but from the equation used it is evident that this sum represents the lengths of the ordinates between axis of x and the curve; to obtain the entire sum, then, must be added to this, that of those equal lengths between the tangent ox and the roadway; this last is evidently $n\delta$, calling δ the vertical distance between the lowest point of the chain and the roadway We have supposed the two branches of the chain symmetric hitherto, but if, in fact, the two points $H\,H'$ were not on the same level, it would



still be easy to determine the point O, having given $A\,A'$ and h and h'; after which each branch could be treated by the preceding method. for calling $A\,A' = L$, we have:

$$L = a + a'$$

also,

$$h = \frac{p}{2Q} a^2$$

and,

$$h' = \frac{p}{2Q} a'^2$$

whence,

$$\frac{h}{h'} = \frac{a^2}{a'^2}$$

or,

$$\frac{a}{a'} = \frac{\sqrt{h}}{\sqrt{h'}}$$

from which values of a and a', we find finally:

$$a = L \frac{\sqrt{h'}}{\sqrt{h} + \sqrt{h'}}$$

and,

$$a' = L \frac{\sqrt{h'}}{\sqrt{h} + \sqrt{h'}}$$

These bridges are very susceptible to the dangers arising from shocks. Suppose a weight $2Q$ to fall from a height h upon the roadway at the middle of a bearer suspended from two rods whose length is l. Let x be the elongation assumed by the rods at the end of a very short time, and suppose a to represent their maximum elongation. Call $2p$ the weight supported by the rods, and ω the section of one.

When then the elongation of the rod has attained its maximum a, the velocity of the falling body having become null, the sum of work developed from the instant of shock by the weight $2Q$ and by the weight $2p$, and by the elasticity of the rods is equal to the living force of the weight $2Q$ at the instant of shock, or,

$$Q(h+a)+pa = \int_0^a \frac{E\omega}{l} x dx = \frac{E\omega a^2}{l}$$

or,

$$Q\left(\frac{h}{l} + \frac{a}{l}\right) + \frac{pa}{l} = E\omega\left(\frac{a}{l}\right)^2$$

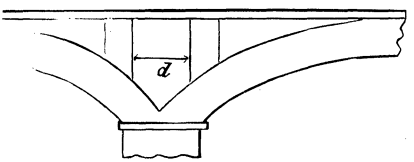
Making $\frac{a}{l} = 0{\cdot}0006$ the proportional elongation beyond which the elasticity of iron is injured, calling also $l = 5^m$; $h = 0^m{\cdot}05$ $\omega = 0^{msq}{\cdot}000001$; $p = 2^{kil}$ and $E = 2 \times 10^{10}$, we shall find :

$$Q = 0^{kil}{\cdot}566$$

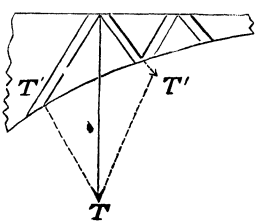
Thus it would suffice that a weight equal to about a quarter of the permanent load should fall on the roadway from a height of 5 centimetres in order to injure the elasticity of the rods.

SPANDREL FILLINGS.

These are usually some upright system of supports. If p represent the uniform weight per linear metre distributed on the roadway, and p' the



test surcharge, and if d be the distance between the uprights, then each will have to support a pressure $T = (p+p')d$. This being substituted in the formula $T = EI\omega$, gives the cross section. If the pieces forming the spandrel filling are oblique, then the load T is decomposed into two, directed

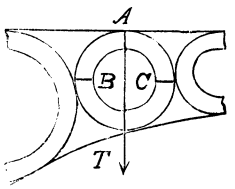


parallel to the axes of the pieces, and these components are used to determine the cross section of each of the pieces. If they are inclined at 45° for instance, we see that

$$T' = T'' = \tfrac{1}{2}T\sqrt{2} = 0.707T$$

The spandrels are sometimes filled by rings inscribed between the roadway and the arch. In such case, for simplicity, we may regard each quadrant as

AB, AC as if fixed at B or C and loaded at A with $\frac{1}{2}T$. The maximum moment of flexure is at



B or C, and its value is $\Sigma M = \frac{1}{2}Tr$. Placing this value in the formula $EI = \frac{\Sigma M V}{\mu}$ and calling the section of the ring a rectangle of dimensions b and h, we shall have:

$$EI = \frac{6\Sigma M}{bh^2} = \frac{3Tr}{bh^2} = \frac{3Tr}{kh^3}$$

calling k the relation between b hnd h, whence h is determined,

$$h = \sqrt[3]{\frac{3Tr}{kEI}}$$

RAIL.

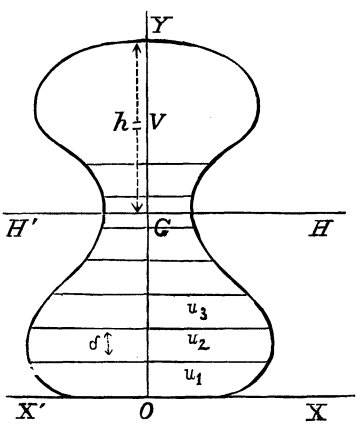
If we regard the rail as first composed of portions fixed at the bearings on the sleepers, and then of independent portions simply resting on these supports, the cross section of the rail may be safely calculated as a mean between the two given by these assumptions. In the first case we shall have:

$$EI = \frac{V}{\mu}\left(\tfrac{1}{8}Pa + \tfrac{1}{12}pa^2\right)$$

and in the second,

$$EI = \frac{V}{\mu}\left(\tfrac{1}{4}Pa + \tfrac{1}{8}pa^2\right)$$

in which p is the weight of the rail per linear metre P the load it is liable to support at its middle point; a, the distance between two sleepers; p then is $p'\omega$, p' being the unit of weight of the rail metal and ω the section of rail; and P is half the weight borne by the axle of the driving wheel of a locomotive augmented by the weight of a driving wheel. To determine ω, V, and μ, the method of quadrature by Simpson's formula may be applied. Assume the desired shape of the cross section and divide its area by equidistant parallels to XX' into an even num-



ber of parts. The distance δ and the lengths u^1, u^2, &c., being measured and known, we have:

$$\omega = \frac{\delta}{3}\Big[4(u_1+u_3+u_5+u_{n-1})+2(u_2+u_4+u_6+u_{n-2})$$

The distance $h = V$ of the centre of gravity from the extreme fibre is:

$$h = \frac{\frac{1}{3}\delta^2\,[4(u_1+3u_3+5u_5+\cdot\cdot)+2(2u_2+4u_4+6u_6+\cdot\cdot)}{\omega}$$

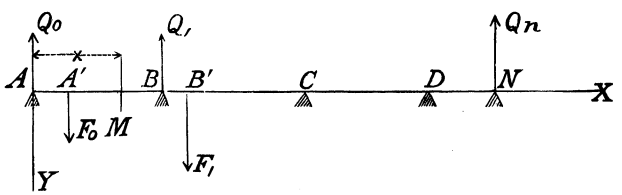
To determine the moment of inertia μ with regard to the axis HH', it will be most convenient to calculate it first with regard to the line XX'. This

moment μ' is given by the formula $\mu' = \int u\,dv\,v^2$. If this integral be replaced by its approximate value, that is, if we take for ordinates the valnes of the products $uv^2 = un^2\delta^2$, we shall find, by Simpson's formula:

$$\mu' = \tfrac{1}{3}\delta^3[4(u_1+9u_3+25u_5+\cdot\cdot)+2(4u_2+16u_4 36u_6+\cdot\cdot$$

Having thus found μ' it is easy to obtain μ, since we always have the relation $\mu' = \mu+\omega h^2$ or, $\mu = \mu'-\omega h^2$. These preliminary calculations being effected, the depth of the rail can be determined as indicated above.

Before discussing longer roof and bridge trusses, it will be necessary to show how to determine the reactions on a beam supported at any number of points and affected by vertical forces. Let a_1, a_2, a_3, &c., be the distances between the successive supports, and let p_1, p_2, p_3, &c., be the weight per



linear metre uniformly distributed on each. Let e be the value of $E\mu$ relative to all the bays, and let us propose to determine the maximum moment of flexure of the whole beam as well as the shape of the curve of the neutral fibre. Call m_0, m_1, m_2, the moments of φ for the different bays at the points A, B, C, &c., and placing the origin first at A, let us consider the equilibrium of the bay AB. If then, as usual, we take the moments $\Sigma MQ = m$ with reference to any point M at the distance x from A, we shall have always $\Sigma MQ =$ a function of x^2 in which the term involving x^2 will be $\frac{1}{2}p_1x^2$. Therefore we may write:

$$\Sigma MQ = e\frac{d^2y}{dx^2} = A + Bx + \tfrac{1}{2}p_1x^2 = m \quad (1)$$

Let now α_0, α_1, α_2, &c., be the inclination of the tangents to the neutral axis at the points A, B, C, we shall have by a first integration of the equation (1):

$$e\left(\frac{dy}{dx} - \alpha_0\right) = Ax + \tfrac{1}{2}Bx^2 + \tfrac{1}{6}p_1x^3$$

and making $x = a$,

$$(2)\ e\left(\frac{dy}{dx} - \alpha_0\right) = Aa_1 + \tfrac{1}{2}Ba^2 + \tfrac{1}{6}p_1a_1^3 = e(\alpha_1 - \alpha_0)$$

Integrating again, we find:

$$e(y - \alpha_0 x) = \tfrac{1}{2}Ax^2 + \tfrac{1}{6}Bx^3 + \tfrac{1}{24}p_1x^4$$

making in this $x = a_1$ which gives $y = 0$, it becomes:

$$-e\alpha_0 a_1 = \text{\&c.},$$

or,

$$(3)\ -e\alpha_0 = \tfrac{1}{2}Aa_1 + \tfrac{1}{6}Ba_1^2 + \tfrac{1}{24}p_1a_1^3$$

Elminating α_0 by subtraction between equations (2) and (3), we have:

$$(4)\ e\alpha_1 = \tfrac{1}{2}Aa_1 + \tfrac{1}{3}Ba_1^2 + \tfrac{1}{8}p_1a_1^3$$

It is evident now that if to discuss the second bay a_2, we change the origin of co-ordinates to B, we shall determine an equation for that bay similar to equation (3), which will be:

$$(5)\ -e\alpha_1 = \tfrac{1}{2}A^1a_2 + \tfrac{1}{6}B^1a_2^2 + \tfrac{1}{24}p_2a_2^3$$

and adding together (5) and (4), we shall find:

$$(6)\ 0 = \tfrac{1}{2}Aa_1 + \tfrac{1}{2}A^1a_2 + \tfrac{1}{3}Ba_1^2 + \tfrac{1}{6}B^1a_2^2 + \tfrac{1}{8}p_1a_1^3 + \tfrac{1}{24}p_2a_2^3$$

But the coefficients A, A^1, B, B^1 can be expressed easily in terms of m_0, m_1, m_2, for from the general value of m in equation (1), we shall have, when $x = 0$,

$$m = A = m_0$$

and when $x = a_1$,

$$m = A + Ba_1 + \tfrac{1}{2}p_1 a_1^2 = m_1$$

whence,

$$A = m_0$$

and,

$$B = \frac{m_1 - m_0}{a_1} - \tfrac{1}{2}p_1 a_1$$

Likewise considering the second span, we should have:

$$A^1 = m_1$$

and,

$$B^1 = \frac{m_2 - m_1}{a_2} - \tfrac{1}{2}p_2 a_2$$

Substituting these in equation (6), we find formula,

$$(M)\ m_0 a_1 + 2m_1(a_1 + a_2) + m_2 a_2 = \tfrac{1}{4}(p_1 a_1^3 + p_2 a_2^3)$$

If the two consecutive bays are equal in length and loaded with the same weight per linear unit, we can write:

$$m_0 + 4m_1 + m_2 = \tfrac{1}{2}pa^2$$

If there are n bays, we shall find, by applying formula M to the bays successively, n—1 equations analagous to equation (M) between the n—1 quantities $m_0 m_1 m_2 \cdots m_n$. But if the beam be simply placed on the supports, the ends, being free, do only experience the reaction of the supports, whence:

$$m_0 = 0$$

and,

$$m_n = 0$$

Therefore there remain $n-1$ equations of the 1st degree to determine the $n-1$ unknown quantities $m_1 \cdots m_{n-1}$. These being known, we know $A_1 B_1$ $A'_1 B'_1$, and as equation (3) gives the inclination α_0, we may find values for $\alpha_2 \alpha_3$, &c., similar to that of α_1 in equation (4). Moreover, for each bay, we shall have an equation involving y and another involving $\frac{dy}{dx}$ at any point, wherefore the whole curve is known. If we would now determine the vertical stresses, let P, P', P'', designate the vertical forces applied to the first bay, and $l\ l' l''$ their distances from the point A. We then shall have :

$$m = \tfrac{1}{2} p(a-x)^2 + P(l-x) + P'(l'-x) \div \cdots + F_1 (a-x) + m_1$$

in which F_1 is the vertical force at B. If we differentiate this equation with regard to x, we see that

$$\frac{dm}{dx} = -[p(a-x) + P + P' + \cdots + F_1)$$

but this second member is simply the whole vertical force acting to shear the beam at the point x, wherefore, any such force is represented by the differential coefficient of the moment of flexure taken with a negative sign, or,

$$F = -\frac{dm}{dx}$$

and from equation (1) we may find, therefore,

$$F = -(B + p_1 x)$$

and also analogous values for F in all the bays from the expressions for m at any point in any one of them. From such values of F we can deduce, as follows, the reactions exercised by the points of

support on the beam. Let A' and B' be two points distant apart a_1, and situated to the right of but very near to the points A and B; and let us consider the equilibrium of that portion of the prism situated between A' and B'. The exterior forces acting upon the portion are its weight $= p_1 a_1$ and the reaction $-Q$, exerted by the support. The molecular forces exerted at A' by the section situated to the left of A' are a vertical force equal and contrary to the vertical or shearing force at A' or to $-F_0$, and a couple whose moment is equal with contrary sign to the moment of flexure m_0. The molecular forces exerted at B' by the section situated to the right of B' likewise reduce to a vertical force which is the shearing force on that section $= +F$, and to a couple whose moment is m. As the sum of the projections of the forces must be equal to 0, we can write, $-F_0 + p_1 a_1 - Q_1 + F_1 = 0$. Since the forces of each couple destroy themselves, and therefore:

$$F_0 - F_1 = p_1 a_1 - Q_1$$

If we call $F_2, F_3, \cdots F_n$ the vertical offorts at points near to and on the right of the points of support, and if we call Q_2, Q_3, &c., the unknown reactions, we shall have for each span an equation analogous to the preceding, and moreover observing that F_n is null since the prism does not extend beyond the last support, we have:

$$F_1 - F_2 = p_2 a_2 - Q_2$$

and,

$$F_{n-1} - F_n = p_n a_n - Q_n$$

or,

$$F_{n-1} = p_n a_n - Q_n$$

These equations, together with the one $Q_0 = F_0$ will make known all the reactions.

As an application of this theory we will take at first the simple case, already treated, of two equal

bays and the same weight distributed on both. We see here that in equation (M) m_0 and m_2 both become zero because the two ends of the beam are free ; wherefore by formula M:

$$4m_1 = \tfrac{1}{2}pa^2$$

or,

$$m_1 = \tfrac{1}{8}pa^2$$

whence,

$$A = 0, \quad B = -\tfrac{3}{8}pa, \quad A^1 = \tfrac{1}{8}pa^2 \quad B^1 = -\tfrac{5}{8}pa$$

whence,

$$\alpha_0 = +\frac{pa^3}{48e}$$

and,

$$\alpha_1 = 0$$

α_2 also will be found equal to α_0 with contrary sign, or to $-\dfrac{pa^3}{48e}$. For $x = 0$, we shall find :

$$F_0 = \tfrac{3}{8}pa, \quad F_1 = \tfrac{5}{8}pa$$

and,

$$Q_0 = \tfrac{3}{8}pa, \quad Q_1 = \tfrac{5}{4}pa, \quad Q_2 = \tfrac{3}{8}pa$$

we should therefore have :

$$e\frac{dy}{dx} = \tfrac{1}{48}pa^3 - \tfrac{3}{16}pax^2 + \tfrac{1}{6}px^3$$

and also,

$$ey = \tfrac{1}{48}pa^3x - \tfrac{1}{16}pax^3 + \tfrac{1}{24}px^4$$

These equations make known the form of the curve of the neutral axis. Equalling $\dfrac{dy}{dx}$ to 0, we obtain

an equation which admits a root $x = a$ already known. Disembarrasing the equation of this, it will remain an equation of the second degree, one of whose roots is comprised between 0 and a, namely: $x = 0.359a$. The ordinate of this point then is maximum, it is the versin of the first bay; its value will be obtained by replacing in the equation which gives y, the variable x by $0.359a$. In the equation $m = -\frac{3}{8}pax + \frac{1}{2}px^2$ we see that the maximum moment of flexure is for $x = a$, and its value is $m = \frac{1}{8}pa$. This is, therefore, the value to put for ΣMQ in the equation of strain.

If we treat, in the same manner, the case of five equal bays, weighted with the same load p, we shall have the four equations:

$$m_0 + 4m_1 + m_2 = \tfrac{1}{2}pa^2$$
$$m_1 + 4m_2 + m_3 = \tfrac{1}{2}pa^2$$
$$m_2 + 4m_3 + m_4 = \tfrac{1}{2}pa^2$$
$$m_3 + 4m_4 + m_5 = \tfrac{1}{2}pa^2$$

in which m_0 and m_5 are each $= 0$. Resolving, we shall find: $m_1 = \frac{4}{38}pa^2$, $m_2 = \frac{3}{38}pa^2$, $m_3 = \frac{3}{38}pa^2$, $m_4 = \frac{4}{38}pa_2$.

Also, $F_0 = \frac{15}{38}pa$, $F_1 = \frac{20}{38}pa$, $F_2 = \frac{19}{38}pa$, $F_3 = \frac{18}{38}pa$, $F_4 = \frac{23}{38}pa$, $F_5 = 0$.

Whence,

$$Q_0 = \tfrac{15}{38}pa$$
$$Q_1 = \tfrac{43}{38}pa$$
$$Q_2 = \tfrac{37}{38}pa$$
$$Q_3 = Q_2$$
$$Q_4 = Q_1$$
$$Q_5 = Q_0$$

The sum of which last values is equal to $5pa$, as should be the case. If we determine the maximum of m, we shall find it to be either m_1 or m_4, which is $\frac{4}{38}pa$, and this then is the value for ΣMQ to be

placed in the equation $El = \frac{\Sigma MQV}{\mu}$. To deduce the dimensions of the beam, we should also find :

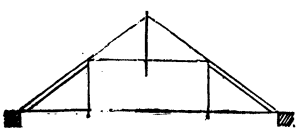
$$a_0 = +\frac{11pa^2}{456E\mu}$$

$$a^1 = -\frac{3pa^2}{456E\mu}$$

$$a^2 = \frac{\nu a^2}{456E\mu}$$

$$a_3 = -a_2, \qquad a_4 = -a_1, \qquad a_5 = -a_0$$

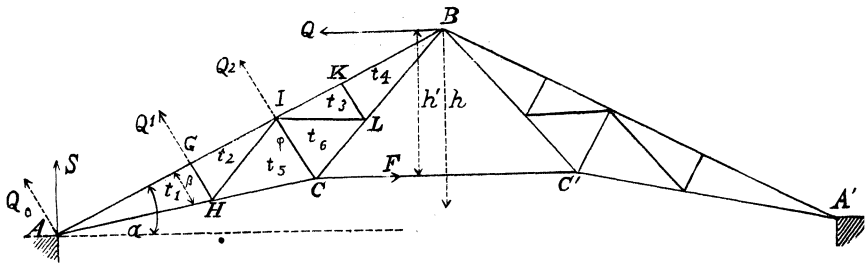
To return now to trusses, let us suppose one furnished with king-post, queen-post and a straining beam. Here we first make abstraction of the posts



and the straining beam and determine, as in the first case, the reactions at the peak of the roof and the tension on the tie-beam. This will give, approximately, the dimensions and the weight of this piece which we then consider as a prism supported at four points whose reactions we calculate. These reactions permit the calculation of the dimensions of the queen-posts and approximately those of the straining beam. Considering the rafter, then, as a beam supported at three points, and knowing about the weight of the straining beam which the king-post supports, the reactions at the points of support of the rafter may be known, and consequently the entire reactions on the frame of the straining beam.

The calculations may be recommenced introducing the weights of the pieces to obtain a closer approximation.

IRON TRUSS.



The rafters in these long span trusses are usually cast iron of double T cross section, the braces of wrought or cast iron, and the ties of wrought iron. To calculate the transversal dimensions of these pieces, their weight must be first neglected; the rafter is supposed in the above figure to be a beam supported by five equidistant reactions and loaded with an uniform weight. Let p be the weight of roofing per linear metre of rafter, and α the angle of AB with the horizon. The weight then perpendicular to the axis of the rafter is $p\cos.\alpha$, and by applying the principles just developed, we may find the reaction at A and $B = Q_0 = \frac{11}{112} pl\cos.\alpha$, at G and $R = Q_1 = \frac{32}{112} pl\cos.\alpha$, and at $I = Q_2 = \frac{26}{112} pl\cos.\alpha$, in which l is the length AB. The tension F is such that the reaction of the wall at A is vertical. The reaction of the rafters at their junction is horizontal, and therefore the equilibrium of a half of the system gives:

$$F = Q, \quad S = pl, \quad pl\tfrac{1}{2}l\cos.\alpha + F(h-h') - Qh = 0$$

which last reduces to

$$\tfrac{1}{2}pl^2\cos.\alpha - Fh' = 0$$

whence,

$$F = \frac{pl^2\cos.\alpha}{2h'}$$

Call t_1, t_2, t_3 the tensions of the ties AH, &c., and call φ the strain on the strut IC, it is evident that seven equations are required to determine these quantities. If therefore, now we consider the equilibrium of the point A under the action of the forces Q_0, S, and t, we must have:

$$-Q_0 + S\cos.\alpha - t_1 \sin.B = 0$$

which can be written,

$$-\tfrac{11}{112}pl\cos.\alpha + pl\cos.\alpha - t_1\sin.B = 0$$

whence,

$$t_1 = \tfrac{101}{112}pl\frac{\cos.\alpha}{\sin.B} \quad (1)$$

Likewise t_4 is the resultant of Q and $-Q_0$, whence,

$$-Q_0 + Q\sin.\alpha - t_4\sin.B = 0 \quad (2)$$

from which t_4 can be obtained, replacing $-Q_0$ and Q by their values. Considering then the equilibrium of the point H under the action of t_1 t_2 t_3 and $-Q_1$ and projecting these forces parallel and perpendicular to AB, we must have:

$$(t_2 + t_5 - t_1)\cos.B = 0$$

or,

$$t_2 + t_5 - t_1 = 0 \quad (3)$$

also,

$$-Q_1 + (t_1 + t_2)\sin.B - t_5\sin.B = 0$$

or,

$$\tfrac{32}{112}pl\cos.\alpha = (t_1 + t_2 - t_5)\sin.B \quad (4)$$

In the same manner, considering the point L, we may find:

$$t_3 + t_6 - t_4 = 0 \quad (5)$$

and,

$$\tfrac{32}{112}pl\cos.\alpha = (t_3 + t_4 - t_6)\sin.B \quad (6)$$

Finally, considering the forces exercised on the strut at the point I, namely, $-Q_2$, t_2, and t_3, we shall have:

$$\varphi = Q_2 + (t_2 + t_3) \sin. B \quad (7)$$

These seven equations will give the required strains, and from them, with the other known strains, the dimensions of all the members of the truss can be determined. The rafter will be calculated from the formula $El = \frac{\Sigma MV}{\mu} - \frac{T}{23}$, in which M is the maximum moment of flexure given by the five reactions at the points of support, and T is the sum of the projections of S and t_1 on the direction of AB, namely:

$$S \sin. a + t_1 \cos. B$$

or,

$$T = pl \sin. a + \tfrac{101}{112} pl \cos. a \frac{\cos. B}{\sin. a}$$

PURLINS.

The purlins which support the common rafters on the principal trussed rafters are considered as solids supported at their ends. If a be the length of a purlin and b the interval of two consecutive ones, and p the weight of roof per square metre, then the total weight supported by a purlin is $2pa\frac{1}{2}b$, or, pab. This is uniformly distributed; the maximum moment of flexure is at the middle, and consequently its expression is:

$$\Sigma M = \tfrac{1}{8} pba^2$$

This value must be placed in the formula $EI = \frac{\Sigma MV}{\mu}$

If the purlin is square, and x be its side, we have:

$$V = \tfrac{1}{2} x; \quad \mu = \tfrac{1}{12} x^4$$

whence,

$$EI = \frac{b\Sigma M}{x^3}$$

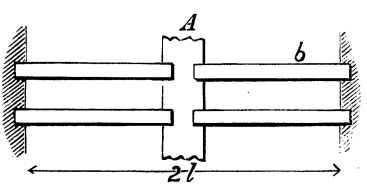
and,

$$x = \sqrt[3]{\frac{3pba^2}{4EI}}$$

in which for oak EI may be 600,000 or 800,000, and for yellow pine, 800,000 or 900,000, according to quality.

FLOORING.

Consider a girder A supporting joists (b) at the middle of an apartment whose length is $2l$. Each joist loads the girder with half the weight it carries, so that, in short, the girder may be supposed uni-



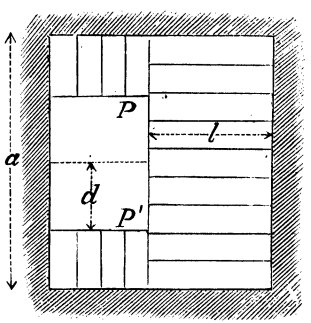
formly loaded with half the weight of the flooring, or with pal if we call a the length of the girder and p the weight of floor per square metre, the maximum moment of flexure is at the middle of the girder, and is expressed by $\Sigma M = pa^2l$. The girders are generally square, and calling the side $= x$, we must have:

$$EI = \frac{\frac{1}{2}x}{\frac{1}{12}x^4}\,\frac{1}{8}pla^2$$

or,

$$x = \sqrt[3]{\frac{3pla^2}{4EI}}$$

If $a = 5^m$, $l = 4^m$, $p = 400^{kg}$, and $EI = 700{,}000$ for oak, then $x = 0^m350$. If the girder be of iron and of a double T cross section, the thickness of the web may be called x and the dimensions of the flanges kx. k being, in practice, assumed, we may may determine x from the formula $EI = \frac{V}{\mu}\frac{1}{8}pla^2$ in which $EI = 6{,}000{,}000$. In the preceding calculation the weight of the beam having been neglected, it is sometimes worth while to determine it from the above approximate dimensions, and introduce it into a new calculation for cross section. But it is to be observed that the formula used was referrible to a beam supported simply at both ends, while the girder is actually more or less fixed at those points, which fact tending to increase its resistance, may be accepted as an offset to the neglect of the beam's weight.



In many cases a girder supports on one side the ends of a number of equidistant joists, but on the other two bearers sustaining each another system of joists. This girder may be supposed loaded uniformly with the weight of the joists and flooring on them equal to $\frac{1}{2}pl$ per linear metre and also subjected to two weights P and P'. These forces P and P' are

each half the weight supported by the bearers, and if they are equal and each applied at the same distance d from the middle of the girder, the maximum moment of flexure will be at the middle of the girder. But the reaction of one wall is one-half the whole weight supported by the girder, or $\frac{1}{2}(\frac{1}{2}pla+2P)$, its moment is then $(\frac{1}{4}pla+P)\frac{a}{2}$, the moment of P is Pd, and the moment of the joists and flooring on one-half the girder is $\frac{1}{4}pla\frac{a}{4}$, whence,

$$\Sigma M = (\tfrac{1}{4}pla+P)\frac{a}{2}-\tfrac{1}{4}pla\frac{a}{4}-Pd$$

or,

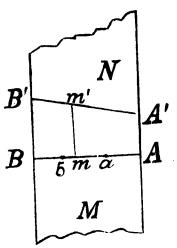
$$\Sigma M = \tfrac{1}{16}pla^2+P\left(\frac{a}{2}-d\right)$$

and if this value be substituted in the formula $EI = \frac{\Sigma MV}{\mu}$ the dimension of the girder may therefrom be determined. As for the joists, if a is the length of one, b the distance apart of all, p the weight of flooring per linear metre, the weight borne by a joist is $2pa\frac{1}{2}b$, or, pb per linear metre. As the maximum moment is at the middle of the joist, we can write $\Sigma M = \frac{1}{8}pba^2$ and determine the section of the joists from the usual formula.

DISTRIBUTION OF PRESSURE ON SURFACES IN CONTACT.

In constructions where prisms are in contact by plane faces and exerting mutual pressure, it would be convenient to know the law according to which

these pressures are distributed at different points. To that end certain hypotheses are made, and such as may be accorded without repugnance, and which, moreover, seem confirmed by experience. We admit that when two solid bodies touch along a plane surface that the conditions are just as if the bodies were composed of fibres normal to the surfaces of contact and that these fibres undergo a shortening proportional to the pressure they experience; also, it may be assumed that the molecules which were in one and the same cross section before compression, remain yet in one and the same section afterward. Let, then, such a face of contact be a plane projected in AB and call e the thickness of the prisms



perpendicular to the plane of the figure. Suppose that by virtue of the pressure between the bodies M and N the section AB of N assumes the position $A'B'$, any fibre is then shortened by a variable decrement such as $m'm$; call a the distance AB, x the distance Am, y the length mm', p the normal pressure per square metre at the point m, and therefore calling K a co-efficient, we may write $p = Ky$; an elementary pressure is consequently $Kyedx$, and the total pressure is $\int_0^a Kyedx$, or $Ke\int_0^a ydx$. But this integral is nothing but the area of the trapezoid $AA'B'B$, therefore calling $AA' = y'$ and $BB' = y''$ the total pressure becomes,

$$Ke\tfrac{1}{2}a(y'+y'')$$

But moreover if P designate the mean pressure per square metre, the total pressure is Pae, wherefore we may write,

$$Pae = Ke\tfrac{1}{2}a(y'+y'')$$

or,

$$P = \tfrac{1}{2}K(y'+y'')$$

The projection of this pressure passes through the centre of gravity of the trapezoid $AA'BB'$; therefore if we call X its distance from AA' we find in considering the moments of the two triangles which compose the trapezoid:

$$X\tfrac{1}{2}a(y'+y'') = \tfrac{1}{2}ay'\tfrac{1}{3}a+\tfrac{1}{2}ay''\tfrac{2}{3}a$$

whence,

$$X = \frac{\tfrac{1}{3}ay'+\tfrac{2}{3}ay''}{y'+y''}$$

calling also p' and p'' the pressures per square metre at A and B, it follows that $p' = Ky'$, and $p'' = Ky''$, and that values for p' and p'' may be found by eliminating y' and y'' by means of the preceding equations; for example, we have,

$$\frac{y'}{y+y''} = \frac{p'}{2P}$$

and also,

$$X = \tfrac{2}{3}a-\tfrac{1}{3}a\frac{y'}{y'+y''}$$

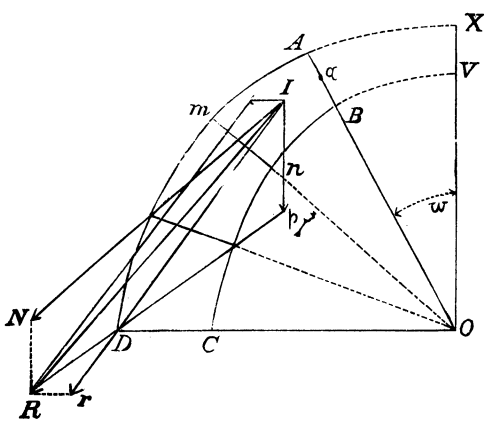
combining these, we find,

$$p' = 2P\left(2-\frac{3X}{a}\right)$$

In the same manner we may place $X = \tfrac{1}{3}a+\tfrac{1}{3}a\dfrac{y''}{y'+y''}$ and $\dfrac{y''}{y'+y''} = \dfrac{p''}{2P}$ whence,

$$p'' = 2P\left(\frac{3X}{a}-1\right)$$

If we make $X = \frac{1}{3}a$, we find $p' = 2P$ and $p'' = 0$, that is, the pressure at A is double the mean pressure, and the pressure at B is nothing. For a value of X inferior to $\frac{1}{3}a$, p'' would have a negative value, that is to say, there would be no bearing at B, a condition which could not be allowed in construction, and therefore it follows that in these cases the resultant pressure must not be permitted to pass nearer to an edge than the third of the length of the joint. It is for this reason that in determining graphically the pressures on the different voussoirs of an arch, the resultant curve of pressures is restricted to pass between the two curves drawn through points of the joints situated at that distance from the edges. The same may be said for the curve of pressures on the joints of a dome though there be conical surfaces. The curve, however, is is not exactly similar to that in cylindrical arches, but may be determined by an extension of the method applied in that foregoing case. Let, for example, $ABCD$ be the section of a portion of dome by a vertical plane through the axis OV. AB, mn, &c., are the traces of the conical joints.



Imagine the dome divided into a number n, of segments by a system of planes near together, passing through the axis OV, and let the figure represent the

mean section of one of these segments. Its equilibrium is as follows. Call P the weight, say of the part $ABXV$ and the superincumbent weight, if there be any, which is not generally the case. P may decomposed into n forces acting at such points as a in the direction aN normal to the joints AB, and if ω be the angle AOX, then the value of aN is $\frac{P}{n}\sin.\omega$. Compare that force then with the weight p of the first voussoir considered. This composition gives a resultant R, but this does not, as in cylindrical arches, represent all the pressures upon the second voussoir, for the vertical faces of the voussoir receive from the adjoining ones pressures which may be supposed normal to the faces, and which pressures, being symmetric, compose into one which is horizontal, and also in the plane of the figure, and which may be represented by Ih, and the other normal to the joint such as Ir. This last will be the pressure upon the second voussoir, and the first will be the resultant of the lateral pressures. The force r composed with the weight of the second voussoir will give a new resultant, which, in like manner, may be decomposed into two, one horizontal and the other normal to the next joint, and so on. Finally, therefore, the conditions of stability are that these resultants r pass not nearer the edges than one-third the length of the joint, and that the surfaces of the joints be ample to support the pressures measured by the intensities Ir, &c.

PLANE FLEXURE IN CURVED PIECES.

Those curved beams which enter into constructions are ordinarily symmetrical with a vertical plane in the direction of their length. The normal sections have their centre of gravity on a curved line

lying in this plane, and such a line may be called the mean fibre. The forces which solicit this beam are ordinarily symmetric also with regard to the plane of symmetry, so rhat they can be reduced to resultants situated in this plane, under their action the fibres all experience a flexure parallel to the plane of symmetry, that is to say, a plane flexure. The normal sections are usually equal throughout the beam, and at all events, for a small portion, the piece may be assimilated to a prism whereto the theory of plane flexure in straight pieces may be applied. Therefore, if we call, as usual, A the area of cross-section, v the distance of the farthest fibre from the neutral fibre, Ei the molecular strain, T the forces normal to the cross sections, ΣM the sum of moments of those forces which bend the beam, referred to an axis through the centre of gravity of any section. μ the moment of inertia of any section about the same axis, and finally V the distance of the neutral fibre from the mean fibre, we can assume the following equations:

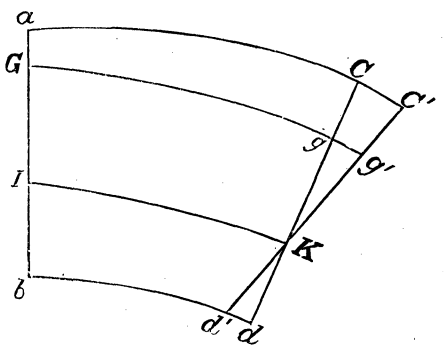
$$(1) \quad T = EiA$$

$$(2) \quad V = \frac{\mu T}{\Sigma M A}$$

$$(3) \quad Ei = \frac{\Sigma M V}{\mu} - \frac{T}{A}$$

In the flexure of pieces originally straight we easily deduced a differential equation, that is, indeed, an approximate one, which being integrated expressed the form of the neutral fibre, but in curved pieces it is necessary to appreciate progressively the effect produced upon the entire beam by the displacement of each one of its normal sections relatively to that which precedes it. In the most ordinary case of curved or arched beam, the piece is not only symmetric with a vertical plane through its axis, but also with another vertical plane perpendicular to the first and passing

through the crown of the beam. After flexure, therefore, in this case, the beam is still symmetric with this last plane and the normal section which it determines can only have moved to rise or fall in the direction of the vertical. This first cause of displacement, if it acted alone, would increase or diminish the vertical ordinates of all points of the neutral fibre, but there are also two other causes of displacement. Consider two sections near together as ab, cd; after flexure cd will have taken the position $c'd'$ turning around some point K in the neutral fibre. But this move-



ment may be considered as the resultant of two others, a movement of translation whereby its centre of gravity g has been transported to g' and a movement of rotation around g'. To measure these two movements, we have,

$$gg' = \lambda$$

but $Gg = l = ds$ calling ds the elementary arc of the mean fibre, therefore as $\frac{\lambda}{ds} = i$, $gg' = ids$ and as $i = \frac{T}{AE}$, we have,

$$gg' = \frac{Tds}{AE}$$

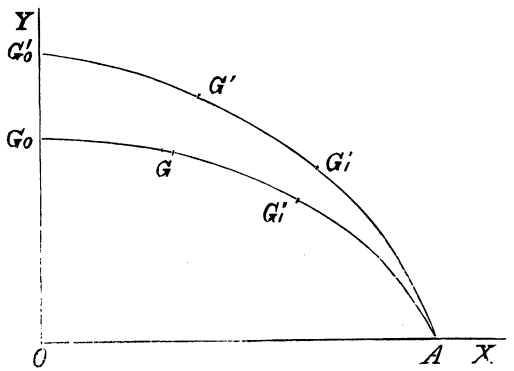
Calling, also, the angle cKc' $d\varphi$, we see that

$$d\varphi = \frac{gg'}{gK} = \frac{ids}{V}$$

or replacing the values of i and V,

$$d\varphi = \frac{\Sigma M ds}{E\mu}$$

From this formula it results that a point situated in the plane of flexure at a distance l from the point g' would describe during the rotation a little arc expressed by $ld\varphi$ or $l\frac{\Sigma M ds}{E\mu}$. That being admitted, let $G_0 G_1$ be any portion of the mean fibre comprised the crown plane and any normal section. Draw the



axes, let f be the ordinate of G_0, x_1, y_1, the co-ordinates of G, and x and y the same for any point of the mean fibre which after flexure shall have taken the position $G_0' G_1'$. The movement from G_0 to G_0' has, firstly, the effect to increase all the ordinates by the quantity $G_0 G_0'$ which call Δf. To appreciate the effect of the second movement, let us consider a normal section at any point G, the element ds which terminates at that point, has lengthened or shortened by the quantity $\frac{Tds}{AE}$, consequently the abscissa of this point has been increased by a quantity equal to the projection of this length on the axis x. But the cosine of the angles which the element ds makes with the axes are $\frac{dx}{ds}$ and $\frac{dy}{ds}$ the projection of the

considered increment or decrement are therefore,

$$\frac{Tdx}{AE} \text{ and } \frac{Tdy}{AE}$$

consequently as by the sole variation of this element the co-ordinates of all the points of the mean fibre of the point G have increased or decreased by the same quantity, it follows that the total variation for the ordinates of any point G_1, is

$$\int_0^{x_1} \frac{Tdx}{AE} \text{ and } \int_0^{y_1} \frac{Tdy}{AE}$$

Now to measure the effect of the second moment, consider again the normal section at G. The rotation of this section around G has caused G_1 to to describe an arc expressed by the formula, if l designates the distance GG_1. But this arc may be considered perpendicular to the right line GG_1, this last making with the axes angles represented by their cosines $\frac{x_1-x}{l}$ and $\frac{y_1-y}{l}$. The cosines moreover of the angles which the arc makes with the same, being $+\frac{y_1-y}{l}$, and $-\frac{x_1-x}{l}$ the corresponding increments or decrements of the co-ordinates x_1 and y_1 are therefore,

$$\frac{\Sigma Mds}{Ei}(y_1-y)$$

and,

$$\frac{\Sigma Mds}{Ei}(x_1-x)$$

whence the entire variation of these ordinates due to the successive rotation of all the normal sections from G_0 to G_1 is expressed by

$$-\int_0^{s_1} \frac{\Sigma Mds}{Ei}(y-y_1)$$

and,

$$+\int_0^{s_1} \frac{\Sigma M ds}{Ei}(x-x_1)$$

If, therefore. we call Δx_1 and Δy_1 the complete variation of the co-ordinates of the point G_1 due to the three movements now described, we will find, making the algebraic summation,

$$\Delta x_1 = \int_0^{x_1} \frac{Tdx}{AE} - \int_0^{s_1} \frac{\Sigma M ds}{E_2}(y-y_1)$$

and,

$$\Delta y_1 = \Delta f + \int_f^{y_1} \frac{Tdy}{AE} + \int_0^{s_1} \frac{\Sigma M ds}{Ei}(x-x_1)$$

These formulas, in which the integrals may be calculated by Simpson's formula, will give for any point its new position after flexure due to T and ΣM. All this presupposes that the forces which act on the piece from G_0 to G_1 are known. If, however, the end A which we may suppose placed on the axis X reposes on a support, we must determine the horizontal and vertical reactions X and y, which are exercised there. To determine y it may be remarked that one can in the equilibrium of the piece suppress the left half of the beam and replace it by a horizontal force applied at G_0^1; therefore, if we represent by P any exterior single force acting on the half $G_0^1 A$, the projections will give, on the vertical,

$$\Sigma P + y = 0$$

or,

$$y = -\Sigma P$$

To determine X, it is evident that as the distance OA is invariable, we must have $\Delta x_1 = 0$, for $x_1 = OA = a$ and $y_1 = 0$. The term which comes from X in T is the projection of X on the tangent at the point x, y. Let α be the angle which this

tangent makes with the axis of X, the term in question is then $-X\cos.\alpha$. The term in ΣM coming from X, is Xy. We may then write the equation:

$$O = \int_0^a \frac{Tdx}{AE} + \int_0^s \frac{\Sigma M ds}{Ei} y - X \int_0^a \frac{dx \cos.\alpha}{AE}$$
$$- X \int_0^s \frac{ds}{Ei} y^2$$

An equation from which X can be drawn, and which value is the thrust which the piece exercises on the support, while y is the load on the same. If, as an illustration, we suppose that the exterior forces are weights proportional to the horizontal projection of the elements of $G_0 A$ and that p is the weight per metre, we will have:

$$T = p(a-x)\sin.\alpha$$

and,

$$\Sigma M = \tfrac{1}{2} p (a-x)^2$$

These values replaced in the expression for X, will allow it to be calculated; y is moreover equal to $-pa$. Δf is determined from the condition that $\Delta y_1 = 0$, for $x_1 = a$ and $y_1 = 0$. When the piece affects a flat curve it is evident that in the value for X the first and third terms are inconsiderable compared to the second and fourth, wherefore, in such case, we might write for the value of X:

$$X = \frac{\int_0^s \Sigma M y ds}{\int_0^s y^2 ds}$$

Finally, if in metallic beams it be desirable to take account of variations of co-ordinates due to variations of temperature, we may be content to vary the values of Δx_1 and Δy_1 by quantities of the form

$t(x_1 - x_0)$ and $t(y_1 - y_0)$ in which t is a numerical coefficient involving the temperature and the coefficient of dilation of the material; it may be written, $t = 0.000012\theta$, in which θ is the increase of temperature.

It might happen that the extremity A were not fixed but able to slide on the axis X, case of a curved beam on two supports, in that condition we should have $X = 0$ and $y = -\Sigma P$. If, then, in the value for Δx_1 we make $x_1 = 0$ and $y_1 = 0$, we find Δa or the variation of the length OA; also, we know that $\Delta y_1 = 0$ for $x_1 = a$ and $y_1 = 0$, which gives the variation of the versin Δf.

An easy graphic method of representing and determining the strain on any portion of a truss, suggested by D. Blanchard, bridgebuilder, may be exhibited as follows:

Since every triangle is half of a parallelogram, it follows that any one represents the equilibrium of three forces, and that therefore a triangle will be in equilibrium under the action of three forces applied perpendicularly to its sides when these forces are in the same proportion as the lengths of the sides to which they are applied. The same is true also of any polygon. This view of the parallelogram of forces is a very good and simple one to direct upon strains in the braces and chords of trusses. Let us, therefore, call such a triangle a triangle of distribution, and present some illustrations of how it should be drawn to determine the dimensions of the various members of trusses.

Let A, Fig. 1, be a simple truss of two braces BB, the junction of which presents a bearing point, represented by the triangle of distribution abc, which is surmounted by the upright column G bearing the weight of W. Then if GB and B are sup-

posed to be of the same dimensions perpendicular to the face of the diagram they will all bear the same strain per unit of cross section in supporting the weight *W*. Thus if *ab* represents the pressure of the weight *W*, then *ac* and *cb* will represent the pressure acting in *B* and *B*. Again, if we suppose the column *G* to be just capable of supporting the weight *W* with safety and economy, and it be desired to support the same weight by two inclined supports spanning the space from *G′* to *G″*, then if we draw the triangle *abc* with the side *ab* equal to the breadth of *G*, and the side *ac* and *cb* at right angles to the inclination of *B* and *B*, and draw *B* and *B* of breadth equal to the length of *ac* and *cb*, and *G*, *B* and *B* are all of equal thickness in the direction perpendicular to the face of the diagram, then *BB* will bear the same strain per unit of cross section as is borne by *G*.

The triangles of distribution at the foot of *B* and *B* serve to show by their perpendicular the comparative amount of thrust at those points, or the strength needed by a tie-beam, if one should be used to restrain it. The base sides of the triangles show the gravity due to *W*, and their sum should be equal to *ab*.

E, Fig. 1, is a modification of A, the only variation is that *B* and *B* stand more inclined to the horizon. This subjects them to greater strain, making an increase in dimensions necessary. The amount of this increase, as well as the increased thrust generated, is delineated in the diagram upon the principle of the triangle of distribution.

I, Fig. 1, also a modification of A, shows the inclined supports in a still more inclined position, subjecting them to still greater strain and a farther increase of thrust. These diagrams are connected by dotted lines, to intimate that the span and supported weight is the same in each, also that the pressure due to gravity at the feet of the inclined

supports is, as it should be, uniform in all. These figures illustrate how the "the triangle of distribution" delineates the strain induced by a given load upon braces at different inclinations. The upright column *G* seldom or never occurs in practical bridge building in the same relation to the braces as is shown in the diagram, but it serves as a graphic representation of the downward pressure to be resolved into two oblique forces.

A, E and I, Fig. 2, are modifications of the same diagrams in Fig. 1, and show how two bearing points are obtained by introducing a straining beam between the upper extremities of the braces. They illustrate the law that governs the relation between the braces and crods of a bridge. The proper dimensions of the straining beam in its relation to the inclined supports is determined by the side *ac* of the triangle of distribution. The volume of thrust and gravity at the foot of the braces is also shown in the figure, the thrust being equal to the force acting in the straining beam, while the gravity is equal to the delineated weight of *W*.

Fig. 3 represents a truss frame of seven bearing points which are obtained by a combination of three different structures, each substantially the same as A, Eig. 2. This truss lacks counter-bracing, and is therefore an incomplete structure for the purpose of a practical bridge; but for supporting a uniformly distributed load, it is theoretically perfect. The braces of each inclination vary in size in the proportion of $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$ and $3\frac{1}{2}$, or assuming a different unit, in the proportion the figures indicate, viz: 5, 15, 25 and 35. The principle of these proportions are too obvious to need detailed explanation, this being the self-evident ratio in which the load accumulates on the braces from the centre toward the end of the bridge.

The braces represented in this figure are called the main braces, because when a bridge is fully and uni-

formly loaded, these alone are brought under strain. They act directly from the abutment toward the centre of the bridge, and therefore should increase in strength by regular additions from the centre to the end of the truss, and these additions should be equivalent to the strain induced by the load pertaining to each bearing point. Each pair of braces of which a truss frame is composed will generate a distinct volume of strain extending the whole distance between their upper extremities; and the sum of all these make up the whole stress upon the upper chord. The action upon the lower chord is exactly the same in principle, though reverse in direction to that upon the upper one. The strain here is tension instead of compression, but the strain between the top and bottom of any two braces is the same in amount. These braces are drawn at an angle of about 55° with the horizon (being the exact angle of the hypotheneus of a rightangle triangle whose base and perpendicular bear the proportion of 6 to 8), but braces of greater or less inclination may be employed. If the braces stand in a more upright position their length will be greater, but the strain upon them and the lateral strain generated by them, less (as indicated by the triangle of distribution). If the braces stand more inclined to the horizon their length will be less, but the strain upon them and upon the chords will be proportionally augmented, as illustrated by A, E and I, Fig. 3.

The inclination to be adopted in construction is usually governed by practical considerations, as the height of the truss, the distance between bearing points, etc. So far as the braces themselves are concerned an angle of 45° in the most economical inclination; but the introduction of the chords complicates the problem.

If a truss frame upon the principle of Fig. 8 is only partially loaded, and this all at one end, it will fail to afford proper support, as there is no series of inclined supports emanating from the opposite abutment reaching the load, and the whole stands, as it

were, upon one leg. This shows the necessity of counter-bracing, or of running a succession of inclined supports from each abutment to each bearing point. Some bridges are constructed upon the idea of a separate support to each point from the end of the structure, which is a simple and effective arrangement of braces, but the more usual method is to start from the original support with an aggregate sufficient for all points and leave to each its supply in passing.

To supply the counterbraces to Fig. 3, we have introduced Fig. 4 which is just adequate to support four of the seven weights of Fig. 3. If we suppose Fig. 4 superimposed upon Fig. 3, we shall readily see by reversing ends, where and to what extent Fig. 3 is deficient in counter-bracing.

The figures upon these diagrams indicate the strain upon each part under the conditions assumed in each, and will readily show where Fig. 4 is in excess of Fig. 3, and therefore what Fig. 3 wants to render it perfect under either condition.

An effective method of resisting the outward thrust at the foot of the braces may be adopted by omitting the tension-chord in short spans or where sufficient resistance can be had at the abutments to counteract the thrust, and interposing thrusting-beams from the abutments, diminishing towards the centre of the bridge, as may be seen by the dotted line in Fig. 3. These, if properly proportioned and skillfully inserted, constitute the simplest and most effective method of resisting the outward lateral thrust at the foot of the braces. This arrangement has also a decisive advantage over the tension-chord in the more satisfactory action of the counter-braces, which in all cases act by compression, and are never relaxed and loosened by the stress of a load, as is the case when the bottom chord acts by tension. They may be said to take off or assume a part of the strain due to the top chord, as each to the extent due to its occupation of the opposite side of the triangle counteracts a part of the lateral thrust generated by the main braces.

Fig. 5 represents an arch proportioned upon the principle of the triangle of distribution. It is, in fact, an extension of A, E and I, Fig. 1, or a modification of Fig. 3. The process of drawing it is simple. Suppose we determine how broad an upright column of the material of which the arch is to be built and commensurate in thickness with that of the arch would be required to support the load pertaining to each voussoir, then make this breadth the measure of the upper side of each triangle of distribution, of which there will be as many, save one, as there are voussoirs in the arch. Distribute them over the projections of the arch with intervening spaces equal to the lateral extension of each voussoir. Then draw vertical lines from the ends of the lines representing the upper sides of the triangles and sufficiently produced downward to cut the upper side of the arch. Then complete the centre triangle by drawing its sides at right angles to the assumed direction of the two centre voussoirs; then draw the voussoirs in the form of a rectangle lying in the assumed direction, and extend their length till the upper corner of the lower end touches the first vertical line of the next bearing point on each side; then draw the upper side of these triangles horizontally and equal to the distance between the vertical lines, and complete these two triangles by connecting the outward end of the upper line to the lower corner of the already completed voussoirs. In like manner draw the next two voussoirs at right angles to this last line, and extend their length as before until the upper corner of the lower end touches the next vertical line, and so proceed through all the bearing points to the haunches of the arch.

In other words, insert between each two voussoirs a triangle of distribution, whose upper side represents the breadth of an upright column capable of supporting the load pertaining one-half of the two adjacent voussoirs, it being scarcely necessary to reiterate that the voussoirs should be drawn of breadth equal and at right angles to the sides of the triangle

to which they adjoin. It will be found that an arch so drawn will have the general outline of the catenary curve and will be an arch of equilibrium, that is, it will have the stability pertaining to perfect balance.

If one part of an arch should require to support a greater weight than another, the correct form will be obtained by giving the triangles of each part proportionate width on top.

If we suppose this arch to be inverted and acting by tension, the law of proportion holds good as to the strain upon the different ports.

On Plates 2, 3, 4, and 5, are presented diagrams showing the fundamental principles of four different arrangements of trusses drawn according to the principles of the triangle of distribution. These are given partly as an illustration of the power of the graphic system of computation and partly as a comparison of different systems of trusses by analysis.

No further explanation will be necessary, as an examination of the Plates will afford a full and clear conception of the subject.

The four bridges analyzed present the general principles of every possible arrangement.

They are generally known as Finks (Plate 2), Ballmore's (Plate 3), Posts' (Plate 4), and Whipple's (Plate 5).

In treating of them we have discarded every element not absolutely necessary to the principle of each, and concerned ourselves only with purely demonstrable mathematics.

Each of these bridges is drawn with eight panels, giving seven bearing points, each surmounted with an assumed weight of 8 units.

The assumed height of each truss is the same as the length of the panels, as this proportion simplifies the problem in each truss and does equal justice or equal injustice to all.

The unit of length upon which the calculations are based is in the small scale drawings $\frac{1}{100}$ of an inch,

while the unit of strain (or width which represents the strain) is $\frac{1}{820}$ of an inch. The details that accompany each truss are enlarged to ten times these dimensions.

From these units the superficial contenst of each part is obtained by multiplication and figured on the small scale drawings.

The Fink and Bollman Trusses require no counterbracing, at least in theory, and none is computed in these estimates.

But each of the other two require such appliances, and they are taken into account.

The method of applying these counterbraces has already been explained; it is briefly to draw first a balanced truss such as is seen on Plate 4 and 5, supporting a weight at each bearing point, also a truss of the same longitudinal dimensions and similar panel arrangement capable of supporting the greatest possible unbalanced load. Then by superimposing the one upon the other, or rather, the strongest elements of each combining to obtain a truss equal to any possible contingency. In this way the counterbracing of the Post and Whipple truss is calculated. On the drawings the parts selected to make in effective combination are worked with a small cross.

The result of the comparison of these four trusses are as follows, using our assumed units of measurement.

	Tension.	Compression	Total.
Post's Balanced Truss,	24,200	24,200	47,400
Effective Combination,	29,400	25,950	53,550
Whipple Bal. Truss,	2,640	26,400	52,800
Effective Combination,	29,600	27,700	57,300
Bollman, . . .	36,400	36,400	72,800
Fink, . . .	3,840	3,840	76,800

We give the strains upon the balanced trusses as an assurance that our calculations are correct, which is shown by the compression and tension in each case being the same.

As will be seen, these computations embrace one-half of each truss, and that generating the greatest amount of strain is proportionately inferior in theory, though this may not strictly follow in practice. As facility of construction and architectual symmetry vary the aggregate excellence, so as to overcome some advantage in mathematical theory.

In these bridges the end post usually employed is left out of the calulation, but will amount to the same in each case, being each 100 units in length by 28 in strain.

We have said that these four bridges embrace the elements of every possible arrangement. This is true essentially.

The Fink and Bollman Trusses admit of no modification, except to reverse them, making the parts now acting by tension do duty by compression, and vice versa, in which case no change in the total amount of strains engendered.

The Post Truss represents fairly every possible arrangement where both struts and ties occupy a diagonal position.

And every case where one or the other stand upright is represented by the Whipple arrangement, if this is what it should most appropriately be called.

Whether the struts stand upright and the ties diagonally or vice versa will make no difference in the amount of strains generated, or if a single or double system is used, the same strains will in the aggregate be engendered.